Primitive line bundles and multiple fibers on holomorphic Lagrangian fibrations

Misha Verbitsky

Estruturas geométricas em variedades,
IMPA, July 24, 2025

(joint work with Ljudmila Kamenova)

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$**,** $\omega_J := g(J\cdot, \cdot)$**,** $\omega_K := g(K\cdot, \cdot)$**.**

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

Hyperkähler manifolds of maximal holonomy

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **maximal holonomy**, or **IHS** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold **admits a finite covering which is a product of a torus and several hyperkähler manifolds of maximal holonomy.**

Further on, **all hyperkähler manifolds are assumed to be of maximal holonomy**.

Lagrangian fibrations

THEOREM: (Matsushita) Let M be hyperkähler manifold of maximal holonomy, and $\pi: M \longrightarrow X$ a surjective holomorphic map, with $0 < \dim X < \dim M$. **Then π is a Lagrangian fibration** (that is, has holomorphic Lagrangian fibers).

THEOREM: (Hwang) In these assumptions, **X is biholomorphic to $\mathbb{C}P^n$ when it is smooth.**

CONJECTURE: X is biholomorphic to $\mathbb{C}P^n$ **when it is normal.**

THEOREM: (Matsushita)
Let M be hyperkähler manifold of maximal holonomy, and $\pi: M \longrightarrow X$ a Lagrangian fibration, with X normal. **Then $H^*(X, \mathbb{Q}) \cong H^*(\mathbb{C}P^n, \mathbb{Q})$.**

REMARK: General fibers of π are Abelian varieties (projective complex tori), by Arnold-Liouville. Conversely, as shown by Hwang-Weiss, **any Lagrangian complex torus in M is a fiber of a Lagrangian fibration.**

Multiplicity of the fibers

THEOREM: (Clemens-Persson)
Let $\pi: M \to X$ be a proper holomorphic map, and F_x its fiber in $x \in X$. **Then there exists a neighbourhood $U \subset X$ of x and a continuous retraction of $\pi^{-1}(U)$ to F_x.** ■

DEFINITION: Let $\pi: M \longrightarrow X$ be a proper holomorphic map, $x \in X$ a point, and $\mathfrak{m}_x$ it maximal ideal. Denote by $F_x := \pi^{-1}(x)$ the set-theoretic preimage of x, understood as a complex variety, and let $F_1, ..., F_k$ be its irreducible components. **(Scheme-theoretic) multiplicity of π in F_i** is the rank of $\mathcal{O}_M/\pi^*\mathfrak{m}_x$ in a general point of F_i.

REMARK: Let $\pi: M \longrightarrow X$ be a proper holomorphic map of complex manifolds, with $\dim X = 1$. Consider a general point z in a multiple fiber F_x. Assume that F_x has multiplicity k in z. **Then for an appropriate neighbourhood U_z of $z \in M$, U_z is locally homeomorphic to $U_F \times \Delta$, where Δ is a disk, U_F a neighbourhood of z in F_x, and $\pi(z_1, t) = t^k$. In particular, π restricted to Δ is a k-sheeted covering.** In these coordinates, the Clemens-Persson map acts on U_z by contracting Δ to the origin.

Multiple fibers

DEFINITION: Let $\pi: M \longrightarrow X$ be a proper holomorphic map, $x \in X$ a point, $F_x := \pi^{-1}(x)$ and F_i its irreducible component, with (scheme-theoretical) multiplicity μ_i. Denote the greatest common divisor of μ_i by μ. A fiber is **multiple** if $\mu > 1$. A fiber F_x is **reduced** if $\mu_i = 1$ for all i. A fiber F_x is **has a reduced component** if $\mu_i = 1$ for at least one i.

DEFINITION: Let $\pi: M \longrightarrow X$ be a surjective holomorphic map of complex manifolds, and $D \subset X$ its set of critical values, which is known as **the discriminant**, or **the discriminant divisor** (it has codimension 1). We say that π **has no multiple fibers in codimension 1** if for a general point $x \in D$, the fiber $\pi^{-1}(x)$ is not multiple.

Multiple fibers of Lagrangian fibrations

Recall that a class $\eta \in H_k(M,\mathbb{Z})$ is called **primitive** if it is not divisible, that is, there is no $\eta' \in H_k(M,\mathbb{Z})$ such that $\eta = r\eta'$, with $r \in \mathbb{Z}$, $|r| \geqslant 2$.

The main result of today's talk

MAIN THEOREM: Let $\pi: M \longrightarrow \mathbb{C}P^n$ be a Lagrangian fibration on a hyperkähler manifold. Let $H \subset \mathbb{C}P^n$ be the hyperplane section. **Then the following assertions are equivalent.**

(i) The homology class of $\pi^{-1}(H)$ is primitive.

(ii) The map π has has no multiple fibers in codimension 1.

(iii) For a general hyperplane section H, the complement $M\backslash\pi^{-1}(H)$ is simply connected.

(iv) The homology map $H^2(M,\mathbb{Z}) \to H^2(\mathbb{C}P^n,\mathbb{Z})$ is surjective.

REMARK: The equivalence (i) $\Leftrightarrow$ (iv) **follows immediately from Poincaré duality.** The implication (i) $\Rightarrow$ (ii) is based on the following theorem, due to R. Thom.

THEOREM: For any orientable smooth n-manifold V, **all elements of the following integral homology groups can be realized by orientable submanifolds:** $H_{n-1}(V,\mathbb{Z})$, $H_{n-2}(V,\mathbb{Z})$, $H_i(V,\mathbb{Z})$ for all $i \leqslant 5$.

Multiple fibers and primitivity

Here we prove the implication (i) $\Rightarrow$ (ii).

PROPOSITION: Let $\pi: M \longrightarrow \mathbb{C}P^n$ be a Lagrangian fibration on a hyperkähler manifold. Let $H \subset \mathbb{C}P^n$ be the hyperplane section. Assume that the homology class of $\pi^{-1}(H)$ is primitive. **Then the map π has has no multiple fibers in codimension 1.**

Proof. Step 1: By Poincaré duality, the homology class of $\pi^{-1}(H)$ is primitive if and only if there exists a homology class $z \in H^2(M, \mathbb{Z})$ such that $z \cap \pi^{-1}(H) = 1$. **By Thom's theorem, this class can be represented by a 2-manifold C.** Using Thom's transversality, we can also assume that C intersects $\pi^{-1}(D)$ transversally in its smooth point, and $\pi(C)$ is smooth.

Step 2: Let $x \in \pi^{-1}(D) \cap C$ be a smooth point on an irreducible component Z_i of multiplicity k. Then $C \cap Z_i$ is divisible by k. Therefore, $\frac{C \cap \pi^{-1}(D)}{\pi(C) \cap D}$ is divisible by the greatest common divisor μ of multiplicities of all Z_i. This is impossible, unless $\mu = 1$, because $[D] = k[H]$, hence $C \cap \pi^{-1}(D) = k = \pi(C) \cap D$. We have proven that **primitivity of $\pi^{-1}(H)$ implies that π has has no multiple fibers in codimension 1.** ■

Fundamental group of $p^{-1}(\mathbb{C}^n)$

THEOREM: Let $p: M \to \mathbb{C}P^n$ be a proper holomorphic surjection of projective manifolds with connected fibers, $\pi_1(M) = 0$, and $H \subset \mathbb{C}P^n$ a generic hyperplane. Let $M_0 := p^{-1}(\mathbb{C}P^n \backslash H)$ **Then $\pi_1(M_0) = 0$ if and only if the homology class $\pi^{-1}(H)$ is primitive.**

Proof: The natural map $\pi_1(M_0) \longrightarrow \pi_1(M)$ is surjective (Chapter IX, Cor. 5.6, SGA1). Its kernel is generated by small loops around the divisor $\pi^{-1}(H)$ which are contracted in a smooth section S of p in a neighbourhood of the divisor $\pi^{-1}(H)$. Consider the long exact sequence of homotopy of a pair

$$\to \pi_2(M) \to \pi_2(M, M_0) \to \pi_1(M_0) \to \pi_1(M) \to 0.$$

The generator of $\ker(\pi_1(M_0)) \longrightarrow \pi_1(M))$ is an element $\tau \in \pi_2(M, M_0)$ represented by the pair $(S, S \cap M_0)$. Since $\pi_1(M, M_0) = 0$ any element of $H_2(M, M_0)$ can be represented by a sphere. This class $\tau \in \pi_2(M, M_0)$ belongs to the image of $\pi_2(M)$ if and only if there is a homology class $\tilde{\tau} \in H_2(M, \mathbb{Z})$ which satisfies $\tilde{\tau} \cap \pi^{-1}(H) = 1$, and this is equivalent to the primitivity of the fundamental class $[\pi^{-1}(H)] \in H^2(M, \mathbb{Z})$. ■

REMARK: To finish the proof of main result, **it remains to show that no multiple fibers in codimension 1 implies the primitivity of $\pi^{-1}(H)$.**

REMARK: So far **all arguments were valid for general proper holomorphic fibrations,** but the last remaining implication is based on hyperkähler geometry.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c>0$ a rational number.**

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\begin{aligned}\lambda q(\eta,\eta) = & \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ & -\frac{n-1}{2n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)\end{aligned}$$

where Ω is the holomorphic symplectic form, and $\lambda>0$.

Remark: **q has signature $(3, b_2-3)$.** It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

Hirzebruch-Riemann-Roch formula

DEFINITION: Let B be a holomorphic vector bundle (or a coherent sheaf). The **holomorphic Euler characteristic** is $\chi(L) := \sum_i (-1)^i H^i(M, B)$.

THEOREM: (Riemann-Roch-Hirzebruch) Let M be a compact complex manifold, and B a holomorphic vector bundle. **Theb $\chi(B)$ can be expressed through the the Chern classes of TM and B,** $\chi(B) = \int_M td(TM) \wedge ch(B)$ where td is the the Todd polynomial on Chern classes of TM,

$$td(M) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^22 - c_4) + ...$$

and $ch(B)$ its Chern character,

$$ch(B) = 1 + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + ...$$

■

Hirzebruch-Riemann-Roch formula and BBF form

THEOREM: (Huybrechts) Let M be a hyperkähler manifold, $\dim_C M = 2n$ and L a holomorphic line bundle. **Then $\chi(L) = \sum a_i q(c_1(L))^i$, where the coefficients a_i are constants depending on the topology of M.**

Proof. Step 1: Let A^* be the subalgebra in cohomology generated by $H^2(M)$. **Then $A^{2i} \cong \operatorname{Sym}^i(H^2(M))$ up to the middle degree, and $A^{n+i} \cong \operatorname{Sym}^{n-i}(H^2(M))$; there is an $O(H^2(M))$-action on cohomology, and the multiplication is $O(H^2(M))$-invariant (V., 1995).**

Step 2: All Chern classes of TM are $O(H^2(M))$-invariant, but there is only one (up to a constant multiplier) $O(H^2(M))$-invariant functional on $\operatorname{Sym}^{2i}(H^2(M))$. On the class $\eta^{2i} \in H^{4i}(M)$ this functional takes value $q(\eta, \eta)^i$. Therefore, **all L-dependent coefficients in the Hirzebruch-Riemann-Roch formula for $\chi(L)$ are expressed through $q(c_1(L))$.** ■

COROLLARY: Let L be a line bundle on a hyperkähler manifold M, $\dim_{\mathbb{C}} M = 2n$. Assume that $q(c_1(L)) = 0$. **Then $\chi(L) = n + 1$.**

Proof: Indeed, $\chi(L) = \chi(\mathcal{O}_M) = n + 1$, with the second equality implied by Bochner's vanishing theorem. ■

Second cohomology of a hyperkähler manifold is torsion-free

CLAIM: Let M be a hyperkähler manifold of maximal holonomy. **Then $H^2(M)$ is torsion-free.**

Proof: The universal coefficients formula gives the exact sequence:

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}(H_1(X;\mathbb{Z}),\mathbb{Z}) \to H^2(X;\mathbb{Z}) \to \operatorname{Hom}_{\mathbb{Z}}(H_2(X;\mathbb{Z}),\mathbb{Z}) \to 0.$$

Since $H_1(X,\mathbb{Z}) = 0$ for a maximal holonomy hyperkähler manifold, **this gives an isomorphism $H^2(X;\mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(H_2(X;\mathbb{Z}),\mathbb{Z})$, hence the torsion vanishes.** ■

ETMDPS vanishing theorem

DEFINITION: A real $(1,1)$-form η on a complex manifold M is called **positive** if $\eta(x, Ix) \geqslant 0$ for all real tangent vectors x.

REMARK: This is a case of so-called **"French positivity"**: in this terminology, 0 is a positive form.

The following theorem was rediscovered several times during 1990-ies (Enoki, Takegoshi, Morougane). Its most general form (which we do not use) is due to Demailly, Peternell and Schneider.

THEOREM: Let (M, I, ω) be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, K its canonical bundle, and L a holomorphic line bundle on M equipped with a Hermitian metric h. Assume that the curvature Θ of L is a positive form on M. Then **the wedge multiplication operator $\eta \longrightarrow \omega^i \wedge \eta$ induces a surjective map**

$$H^0(\Omega^{n-i} M \otimes L) \xrightarrow{\omega^i \wedge \cdot} H^i(K \otimes L).$$

Here ω is considered as an element in $H^1(\Omega^1 M)$, and multiplication by ω maps $H^k(\Omega^{n-l} M \otimes L)$ to $H^{k+1}(\Omega^{n-l+1} M \otimes L)$.

Primitivity and vanishing of cohomology

THEOREM: Let M be a hyperkähler manifold admitting a Lagrangian fibration $\pi: M \to X$, and H a line bundle on X. Let L be a line bundle such that $L^{\otimes k} = \pi^* H$. **Then L is trivial on all smooth fibers of π.**

Proof. Step 1: Let F be a smooth fiber of π, which is an abelian variety by Arnol'd-Liouville. Then $T^*M|_F$ is an extension of a trivial bundle TF with another trivial bundle $NF = T^*F$. **For any non-trivial line bundle $L \in \operatorname{Pic}_0(F)$, we have $H^0(L \otimes TF) = 0$ and $H^0(L \otimes NF) = 0$, which implies that $H^0(L \otimes T^*M|_F) = 0$.** Similarly one $H^0(L \otimes \Lambda^k T^*M|_F) = 0$ (Lemma 2).

Step 2: Unless L is trivial on F, we have $H^0(L \otimes \Lambda^k T^*M|_F) = 0$, which implies that $H^0(L \otimes \Lambda^* M) = 0$. By Enoki-Mourugane-Takegoshi-Demailly-Peternell-Schneider theorem, **this implies that $H^i(L) = 0$, hence $\chi(L) = 0$, contradicting the formula $\chi(L) = n + 1$ (Huybrechts).** ∎

Fiberwise monodromy of a line bundle

Proposition 1: Let M be a hyperkähler manifold admitting a Lagrangian fibration $\pi : M \to X$, and H a line bundle on X. Let L be a line bundle such that $L^{\otimes k} = \pi^* H$. **Then L admits a connection ∇ which is flat on each restriction $L|_F$ to the fiber of π.**

Proof: Choose a constant metric h^k on $L^{\otimes k}|_F = \mathcal{O}_F$ and let h be its k-th root, which is a metric on $L|_F$. **Since h^k is constant, its curvature is flat, and the Chern connection ∇ associated with h is also flat.**

DEFINITION: Fiberwise monodromy of L is its monodromy on the fibers of π.

Remark 1: Clearly, **$L = \pi^* L_0$ if the fiberwise monodromy of L on each fiber is trivial.**

Fiberwise monodromy and the fundamental group $\pi^{-1}(\mathbb{C}^n)$

Proposition 2: Let $\pi : M \to \mathbb{C}P^n$ be a Lagrangian fibration. Consider a general hyperplane section $H \subset \mathbb{C}P^n$. Assume that $\pi_1(M\backslash\pi^{-1}(H)) = 0$. **Then $\pi^{-1}(H)$ is primitive.**

Proof. Step 1: Let k be the maximal integer divisor of $[\pi^{-1}(H)] \in H^{4n-2}(M, \mathbb{Z})$, and L a line bundle such that $L^k = \pi^*\mathcal{O}(1)$. To show that $k = 1$, **it suffices to show that the fiberwise monodromy of L on each fiber of π is trivial (Remark 1).**

Step 2: Let ∇ be the standard connection on $\mathcal{O}(1)$, with its curvature equal to the Fubini-Study form ω_{FS}, and ∇_L the corresponding connection on L (there is a bijection between connections on a line bundle and its tensor powers). Then its curvature is equal to $\frac{1}{k}\pi^*(\omega_{FS})$. On $\mathbb{C}P^n\backslash H$, the form ω_{FS} is exact, moreover, $\omega_{FS} = d\theta$ for some $\theta = d^c f$, where f is the Kähler potential of $\omega_{FS}|_{\mathbb{C}^n}$, $f(z) = const \cdot \log(|z|^2/(1+|z|^2))$. Consider the connection $\nabla_L - \frac{1}{k}\pi^*\theta$ on $L\Big|_{\mathbb{C}P^n\backslash H}$. This connection is by construction flat, and **its fiberwise monodromy is equal to the fiberwise monodromy of ∇_L.**

Step 3: Monodromy of a flat connection on simply connected manifold is trivial. Therefore, $\pi_1(M\backslash\pi^{-1}(H)) = 0$ implies that the monodromy of $\nabla_L - \frac{1}{k}\pi^*\theta$ and the fiberwise monodromy of ∇_L vanishes. ■

Clemens-Persson retraction theorem

THEOREM: (Clemens-Persson)
Let $\pi: M \to X$ be a proper holomorphic map, and F_x its fiber in $x \in X$. **Then there exists a neighbourhood $U \subset X$ of x and a continuous retraction of $\pi^{-1}(U)$ to F_x.** ■

THEOREM: Let M be a hyperkähler manifold admitting a Lagrangian fibration $\pi: M \to X$, and H a line bundle on X. Let L be a line bundle such that $L^{\otimes k} = \pi^* H$. Given a special fiber F_x, consider the Clemens-Persson retraction map $\Phi: \pi^{-1}(U) \longrightarrow F_x$. Let F be a smooth fiber of π. Assume that $\Phi: F \longrightarrow F_x$ is surjective on fundamental groups. **Then L is trivial on F_x.**

Proof: Let $\gamma_x \in \pi_1(F_x)$ be a loop, and $\gamma \in \pi_1(F)$ be a loop such that $\Phi(\gamma) = \gamma_x$. Consider the homotopy Φ_t contracting Id to Φ, with $\Phi_0 = \mathrm{Id}$ and $\Phi_1 = \Phi$, and denote by μ_t the monodromy of the connection ∇ (Proposition 1) along $\Phi_t(\gamma)$. By construction, μ_t is continuous in t, and trivial for all $t \neq 1$, hence **the monodromy of ∇ along γ is trivial, and L is trivial on fibers of π by Remark 1.** ■

Surjectivity in codimension 1

DEFINITION: Recall that a line bundle H is called **primitive** if H is not isomorphic to a non-trivial tensor power of another line bundle.

Theorem 3: Let M be a hyperkähler manifold admitting a Lagrangian fibration $\pi : M \to X$, and H a primitive line bundle on X. Let $D \subset X$ be **the discriminant** of π, that is, the set of singular values of π. Assume that for a general $x \in D$, the Clemens-Persson map $\Phi : F \longrightarrow F_x$ is surjective on fundamental groups. **Then $\pi^* H$ is primitive.**

Proof: Consider a primitive line bundle L on M such that $\pi^* H = L^{\otimes k}$. Let $D_0 \subset D$ be the set of all $x \in D$ such that $L|_{F_x}$ is trivial. This set is Zariski open, hence **its complement $D_1 \subset D$ has codimension 1 in D and codimension 2 in X.**

Step 2: Let $Z := \pi^{-1}(D_1)$, and $j : M \backslash Z \longrightarrow M$ the open embedding. **Then $L = j_* j^* L$ by "Serre's condition S2" (OSS, Ch. II, Lemma 1.1.12)**. This immediately implies that L is a pullback of a bundle L_1 on X.

Step 3: Indeed, the bundle $j^* L$ is trivial on the fibers of π, hence it is obtained as a pullback of a line bundle L_0 on $X \backslash D_1$. By the same argument, $\operatorname{Pic}(X) = \operatorname{Pic}(X \backslash D_1)$, hence L_0 is a restriction of a line bundle L_1 on X. Then $L\Big|_{M \backslash Z} = \pi^* L_1\Big|_{M \backslash Z}$. **Applying "Serre's condition S2" again, we obtain that $L = \pi^* L_1$.** ■

Surjectivity on second homotopy groups and multiple fibers

REMARK: To finish the proof of main theorem, we need to show that the condition “no multiple fibers in codimension 1” implies that the Clemens-Persson map is surjective on fundamental groups; then Theorem 3 will imply that $\pi^{-1}(H)$ is primitive.

Claim 3: Let $p: M \longrightarrow \mathbb{C}P^1$ be a proper holomorphic surjection, and $z \in \mathbb{C}P^1$ a singular value. Denote by M_0 the complement $M \backslash p^{-1}(z)$. Then the natural map $\pi_2(M, M_0) \to \pi_2(\mathbb{C}P^1, \mathbb{C}P^1 \backslash z)$ **is surjective if and only if z is not a multiple fiber.**

Proof: Suppose that the components $D_1, ..., D_n$ of the divisor $D = p^{-1}(z)$ have multiplicity $\mu_1, ..., \mu_n$. The image of $\pi_2(M, M_0)$ in $\pi_2(\mathbb{C}P^1) = \pi_2(\mathbb{C}P^1, \mathbb{C}P^1 \backslash z)$ is generated by sections of p transversal to $D_1, ..., D_n$; each of such sections meets D_i with multiplicity μ_i. ■

The proof of Main Theorem

THEOREM: Let $\pi : M \to \mathbb{C}P^n$ be a Lagrangian fibration, with no multiple fibers in codimension 1. Then **the Clemens-Persson map is surjective on fundamental groups** for general fibers over the points of the discriminant.

Proof. Step 1: Let U be a neighbourhood of the special fiber $\pi^{-1}(z)$ in $\pi^{-1}(\mathbb{C}P^1)$, where $z \in \mathbb{C}P^1 \subset \mathbb{C}P^n$ is a general point of the discriminant, and $U_0 := U \backslash \pi^{-1}(z)$. Then $\pi_1(U_0)$ surjectively maps to $\pi_1(\pi(U)\backslash z)$. Let F be a general fiber of π and consider the long exact sequence of homotopy of a fibration

$$0 \to \pi_1(F) \to \pi_1(U_0) \to \pi_1(\pi(U)\backslash z) \to 0. \quad (*)$$

Since $\pi_1(U_0)$ surjectively maps to $\pi_1(\pi(U)\backslash z) = \mathbb{Z}$, there exists $A \in \pi_1(U_0)$ which maps to the generator of the $\pi_1(\pi(U)\backslash z)$. Then the long exact sequence (*) implies that **the group $\pi_1(U_0)$ is generated by $\pi_1(F)$ and A.**

Step 2: Using Claim 3, we can chose A in such a way that its image inder the Clemens-Persson map vanishes. Then $\pi_1(U)$ is an image of $\pi_1(F)$. ■