Hyperkähler reduction and the moduli of flat bundles on complex curves

Misha Verbitsky

February 4, Thursday, 2016,

Seminar on geometric structures on manifolds,

LAG HSE.

Plan

1. Symplectic reduction and GIT

2. Hyperkähler reduction

3. Moment map for the gauge group action.

Conventions: Further on, G is a compact, connected Lie group, $G_{\mathbb{C}}$ its complexification, $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ the corresponding Lie algebras. **Central element** of $\mathfrak{g}^*$ is one which is fixed by the adjoint action of G.

Cartan's formula and symplectomorphisms

We denote the Lie derivative along a vector field as $\text{Lie}_x : \Lambda^i M \longrightarrow \Lambda^i M$, and contraction with a vector field by $i_x : \Lambda^i M \longrightarrow \Lambda^{i-1} M$.

Cartan's formula: $d \circ i_x + i_x \circ d = \text{Lie}_x$.

REMARK: Let (M, ω) be a symplectic manifold, G a Lie group acting on M by symplectomorphisms, and $\mathfrak{g}$ its Lie algebra. For any $g \in \mathfrak{g}$, denote by ρ_g the corresponding vector field. Then $\text{Lie}_{\rho_g} \omega = 0$, giving $d(i_{\rho_g}(\omega)) = 0$. **We obtain that $i_{\rho_g}(\omega)$ is closed, for any $g \in \mathfrak{g}$.**

DEFINITION: A Hamiltonian of $g \in \mathfrak{g}$ is a function h on M such that $dh = i_{\rho_g}(\omega)$.

Moment maps

DEFINITION: (M,ω) be a symplectic manifold, G a Lie group acting on M by symplectomorphisms. **A moment map** μ of this action is a linear map $\mathfrak{g} \longrightarrow C^\infty M$ associating to each $g \in G$ its Hamiltonian.

REMARK: It is more convenient to consider μ as an element of $\mathfrak{g}^* \otimes_{\mathbb{R}} C^\infty M$, or (and this is most standard) **as a function with values in $\mathfrak{g}^*$**.

REMARK: Moment map **always exists** if M is simply connected.

DEFINITION: A moment map $M \longrightarrow \mathfrak{g}^*$ is called **equivariant** if it is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$.

REMARK: $M \xrightarrow{\mu} \mathfrak{g}^*$ is a moment map iff for all $g \in \mathfrak{g}$, $\langle d\mu, g\rangle = i_{\rho_g}(\omega)$. Therefore, **a moment map is defined up to a constant $\mathfrak{g}^*$-valued function.** An equivariant moment map is is defined up to **a constant $\mathfrak{g}^*$-valued function which is G-invariant**, that is, up to addition of a central vector $c \in \mathfrak{g}^*$.

CLAIM: An equivariant moment map exists whenever $H^1(G,\mathfrak{g}^*) = 0$. In particular, when G is reductive and M is simply connected, an equivariant moment map exists. Further on, all moment maps will be tacitly considered equivariant.

Weinstein-Marsden theorem

DEFINITION: (Weinstein-Marsden) (M,ω) be a symplectic manifold, G a compact Lie group acting on M by symplectomorphisms, $M \xrightarrow{\mu} \mathfrak{g}^*$ an equivariant moment map, and $c \in \mathfrak{g}^*$ a central element. The quotient $\mu^{-1}(c)/G$ is called **symplectic reduction** of M, denoted by $M/\!\!/G$.

CLAIM: The symplectic quotient $M/\!\!/G$ is a symplectic manifold of dimension $\dim M - 2\dim G$.

Proof. Step 1: $T_x(\mu^{-1}(c)) = d\mu^{-1}(0)$, however, $d\mu$ is ω-dual to the space $\tau(\mathfrak{g})$ of vector fields tangent to the G-action, **hence $d\mu^{-1}(0) = \tau(\mathfrak{g})^\perp$.**

Step 2: Since μ is G-equivariant, G preserves $\mu^{-1}(c)$, hence $\tau(\mathfrak{g}) \subset d\mu^{-1}(0)$. This implies that $\tau(\mathfrak{g}) \subset TM$ **is isotropic** (that is, $\omega\big|_{\tau(\mathfrak{g})} = 0$.) Its ω-orthogonal complement in T_xM is $T_x(\mu^{-1}(c))$ (Step 1).

Step 3: Consider the **characteristic foliation** $\mathcal{F}$ on $\mu^{-1}(c)$, that is, the set of all $v \in T_x(\mu^{-1}(c))$ such that $\omega(v,w) = 0$ for all $w \in T_x(\mu^{-1}(c))$ From Step 2 **we obtain that $\mathcal{F} = \tau(\mathfrak{g})$.**

Step 4: Since $\omega\big|_{\mu^{-1}(c)}$ is closed, it satisfies $\operatorname{Lie}_v(\omega) = 0$ for all $v \in \mathcal{F}$. This implies that it is lifted from the leaf space of characteristic foliation, identified with $M/\!\!/G$. ■

Symplectic reduction and GIT

THEOREM: Let (M, I, ω) be a Kähler manifold, $G_{\mathbb{C}}$ a complex reductive Lie group acting on M by holomorphic automorphisms, and G its compact form acting isometrically. **Then $M /\!\!/ G$ is a Kähler manifold.**

Proof: Since the orbits of the $G_{\mathbb{C}}$-action are complex subvarieties, they are symplectic. Since the orbits of $G \subset G_{\mathbb{C}}$ are isotropic, and their dimension is half of dimension of orbits of $G_{\mathbb{C}}$, they are actially Lagrangian subvarieties in orbits of $G_{\mathbb{C}}$. Therefore, $\mu^{-1}(c)$ intersects each orbit of $G_{\mathbb{C}}$ in a G-orbit. **We have identified $M /\!\!/ G$ with a space of $G_{\mathbb{C}}$-orbits which intersect $\mu^{-1}(c)$.** ■

REMARK: In such a situation, $M /\!\!/ G$ is called **the Kähler quotient**, or **GIT quotient**. The choice of a central element $c \in \mathfrak{g}^*$ is known as a choice of **stability data**.

REMARK: **The points of $M /\!\!/ G$ are in bijective correspondence with the orbits of $G_{\mathbb{C}}$ which intersect $\mu^{-1}(c)$.** Such orbits are called **polystable**, and the intersection of a $G_{\mathbb{C}}$-orbit with $\mu^{-1}(c)$ is a G-orbit.

Kähler reduction and a Kähler potential

DEFINITION: Kähler potential on a Kähler manifold (M, ω) is a function ψ such that $dd^c\psi = \omega$.

PROPOSITION: Let G be a real Lie group acting on a Kähler manifold M by holomorphic isometries, and ψ be a G-invariant Kähler potential. **Then the moment map $\mathfrak{g} \times M \stackrel{\mu_g}{\longrightarrow} \mathbb{R}$ can be written as $g, m \longrightarrow -\operatorname{Lie}_{Iv} \psi$,** where $v = \tau(g) \in TM$ is the tangent vector field associated with $g \in \mathfrak{g}$.

Proof: Since ψ is G-invariant, and I is G-invariant, we have $0 = \operatorname{Lie}_v d^c\psi = (dd^c\psi) \lrcorner v + d(\langle d^c\psi, v\rangle)$. Using $\omega = dd^c\psi$, we rewrite this equation as $\omega \lrcorner v = -d(\langle d^c\psi, v\rangle)$, giving an equation for the moment map $\mu_g = -\langle d^c\psi, v\rangle$. Acting by I on both sides, we obtain $\mu_g = -\langle d\psi, Iv\rangle = -\operatorname{Lie}_{Iv} \psi$. ■

COROLLARY: Let V be a Hermitian representation of a compact Lie group G. **Then the corresponding moment map can be written as $\mu_g(v) = -\operatorname{Lie}_{Ig} |v|^2 = -\frac{1}{2}\langle v, Ig(v)\rangle$.** ■

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot,\cdot)$, $\omega_J := g(J\cdot,\cdot)$, $\omega_K := g(K\cdot,\cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel transport along the connection preserves I, J, K.

REMARK:
The form $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is **holomorphic and symplectic** on (M, I).

Hyperkähler reduction

DEFINITION: Let G be a compact Lie group, ρ its action on a hyperkähler manifold M by hyperkähler isometries, and $\mathfrak{g}^*$ a dual space to its Lie algebra. **A hyperkähler moment map** is a G-equivariant smooth map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ such that $\langle \mu_i(v), g\rangle = \omega_i(v, d\rho(g))$, for every $v \in TM$, $g \in \mathfrak{g}$ and $i = 1, 2, 3$, where ω_i is one three Kähler forms associated with the hyperkähler structure.

DEFINITION: Let ξ_1, ξ_2, ξ_3 be three G-invariant vectors in $\mathfrak{g}^*$. The quotient manifold $M /\!\!/\!\!/ G := \mu^{-1}(\xi_1, \xi_2, \xi_3)/G$ is called **the hyperkähler quotient** of M.

THEOREM: (Hitchin, Karlhede, Lindström, Roček)
The quotient $M /\!\!/\!\!/ G$ is hyperkaehler.

Holomorphic moment map

Let $\Omega := \omega_J + \sqrt{-1}\omega_K$. This is a holomorphic symplectic (2,0)-form on (M, I).

The proof of HKLR theorem. Step 1: Let μ_J, μ_K be the moment map associated with ω_J, ω_K, and $\mu_{\mathbb{C}} := \mu_J + \sqrt{-1}\,\mu_K$. Then $\langle d\mu_{\mathbb{C}}, g\rangle = i_{\rho_g}(\Omega)$. Therefore, $d\mu_{\mathbb{C}} \in \Lambda^{1,0}(M, I) \otimes \mathfrak{g}^*$.

Step 2: This implies that the map $\mu_{\mathbb{C}}$ is holomorphic. It is called **the holomorphic moment map.**

Step 3: By definition, $M /\!\!/\!\!/ G = \mu_{\mathbb{C}}^{-1}(c) /\!\!/ G$, where $c \in \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}$ is a central element. **This is a Kähler manifold,** because it is a Kähler quotient of a Kähler manifold.

Step 4: We obtain 3 complex structures I, J, K on the hyperkähler quotient $M /\!\!/\!\!/ G$. **They are compatible in the usual way** (an easy exercise). ■

Gauge group

DEFINITION: Let G be a Lie group, and P a principal G-bundle on M. **The gauge group** of P is the group of G-invariant automorphisms of P.

REMARK: Let $G_{\mathbb{C}}$ be a complex Lie group, P a principal $G_{\mathbb{C}}$-bundle, $\mathcal{A}$ the space of all $G_{\mathbb{C}}$-invariant connections on P, and $\mathfrak{P}_{\mathbb{C}}$ the bundle of G-invariant vector fields tangent to the fibers of P. **Then $\mathcal{A}$ is a complex affine vector space with linearization $\Lambda^1 M \otimes \mathfrak{P}_{\mathbb{C}}$.** It is equipped with the gauge group action.

DEFINITION: Let P_G be a reduction of the principal $G_{\mathbb{C}}$-bundle P to G, and $g \longrightarrow g^t$ be the corresponding real structure operator on $\mathfrak{P}_{\mathbb{C}}$. Then Aut$(P_G)$ is called **the real gauge group**, and Aut(P) **the complex gauge group**. The Killing form on $\mathfrak{P}_{\mathbb{C}}$ is denoted as $a, b \longrightarrow \operatorname{Tr}(ab) \in C^\infty M$.

Hyperkähler structure on the space of connections

CLAIM: Let $P_{\mathbb{C}}$ be a complexification of a principal G-bundle P on a compact Riemann surface M, and $\nabla \in \mathcal{A}$ a connecion in $P_{\mathbb{C}}$. Denote by $g \longrightarrow g^t$ the real involution on $\mathfrak{P}_{\mathbb{C}}$ fixing $\mathfrak{P}$. **Then the tangent space $T_\nabla\mathcal{A}$ is equipped with a real gauge invariant Hermitian form $g(a,b) = \int_M \operatorname{Re}\operatorname{Tr}(a\wedge b^t)$ and a complex linear 2-form $\Omega(u,v) := \int_M \operatorname{Tr}(u\wedge v)$.** ■

DEFINITION: Define quaternionic structure on $T_\nabla\mathcal{A}$ as follows. The complex structure I comes from the complex structure on $\mathfrak{P}_{\mathbb{C}}$, and J comes from $\operatorname{Re}\Omega(x, Jy) = g(a,b)$.

REMARK: For $\lambda\otimes a \in T_\nabla\mathcal{A} = \Lambda^1(M)\otimes_{\mathbb{C}}\mathfrak{g}_C$, we can write $J(\lambda\otimes a) = I_M(\lambda)\otimes a^t$, where I_M is a complex structure operator on M acting on $\Lambda^1(M)$.

COROLLARY: The manifold $\mathcal{A}$ **is equipped with a natural real gauge invariant flat hyperkähler structure.** ■

Hyperkähler moment map

REMARK: The tangent space to the gauge group $\operatorname{Aut}(P)$ can be identified with $\mathfrak{P}_{\mathbb{C}}$. Therefore, **the gauge moment map takes values in $\mathfrak{P}_{\mathbb{C}}^* = \mathfrak{P}_{\mathbb{C}}$.**

THEOREM: Let (M, I, ω) be a Riemannian surface equipped with a Hermitian form ω, P a principal $G_{\mathbb{C}}$-bundle, $\mathcal{A}$ the space of connections on P, and $G \subset G_{\mathbb{C}}$. a compact real form. Consider the hyperkähler structure on $\mathcal{A}$ defined above. **Then the holomorphic moment map associated with the gauge action can be written as $\nabla \longrightarrow -\frac{\Theta_\nabla}{\omega}$, where $\Theta_\nabla \in \Lambda^2 M \otimes \mathfrak{P}_{\mathbb{C}}$ is the curvature of ∇.**

Proof. Step 1: If $\nabla_1 = \nabla + A$, we have $\theta_{\nabla_1} = \Theta_\nabla + \nabla(A) + A \wedge A$. Therefore, differential of Θ_∇ takes $A \in \Lambda^1 M \otimes \mathfrak{P}_{\mathbb{C}}$ to $\int_M \nabla(A)$. If we pair this to $b \in \mathfrak{P}_{\mathbb{C}}$, we obtain $A \longrightarrow \int_M \operatorname{Tr}(b\nabla(A))$. **This is the differential of a moment map evaluated in ∇, b.**

Step 2: The holomorphic symplectic form on $\mathcal{A}$ is expressed as $\Omega(A, B) = \int_M \operatorname{Tr}(A \wedge B)$. For any $b \in \mathfrak{P}_{\mathbb{C}}$, the corresponding vector field on $\mathcal{A}$ is written as $\nabla(b)$, and the corresponding 1-form as $\Omega(\nabla(b), A) = \int_M \operatorname{Tr}(\nabla(b) \wedge A)$. Here $A \in T\mathcal{A}$ is a tangent vector, and $\Omega(\nabla(b), \cdot)$ is considered as a 1-form on $\mathcal{A}$.

Step 3: It remains to compare the 1-forms obtained in Step 1 and Step 2: $\int_M \operatorname{Tr}(\nabla(b) \wedge A) = -\int_M \operatorname{Tr}(b\nabla(A))$. ■

Space of flat connections

THEOREM: The space of stable flat connections on a compact Riemann surface up to (complex) gauge equivalence **is obtained as $\mathcal{M} = \mathcal{A}/\!/\!/\mathcal{G}$,** where $\mathcal{G}$ is the real gauge group. **In particular, $\mathcal{M}$ is hyperkähler.**

Proof: Denote the space of flat connections by $\mathcal{A}_{\mathsf{fl}} \subset \mathcal{A}$. Flatness is vanishing of the holomorphic moment map, hence $\mathcal{A}_{\mathsf{fl}}/\!/G = \mathcal{A}/\!/\!/\mathcal{G}$. However, $\mathcal{A}_{\mathsf{fl}}/\!/G$ is precisely the space of stable flat connections up to complex gauge equivalence. ■

Addendum: Real moment map

PROPOSITION: Since the Hermitian form on $\mathcal{A}$ is fixed by affine transforms, **it can be written as dd^c of a quadratic function which is its Kähler potential. The latter can be written as $\psi(\nabla) = \int_M \operatorname{Tr}[\nabla, \nabla^t]$** (here ∇^t is a connection operator which is real conjugate to ∇).

Proof: Second derivative of this quadratic map is $\operatorname{Hess}(x, y) = \int_M \operatorname{Re}(x \wedge y^t)$.
■

PROPOSITION: The **real moment map for gauge action on the space of connections is $\frac{[\nabla, \nabla^t]}{\omega}$.**

Proof: The gauge Lie algebra action is written as $g(\nabla) = \nabla + \nabla(g)$, for all $g \in \mathfrak{P}_{\mathbb{C}}$. This gives an expression for the real moment map:

$$\mu_g(\nabla) = \frac{d}{dg} \int_M \operatorname{Tr}[\nabla + \sqrt{-1}\,\nabla(g), \nabla^t - \sqrt{-1}\,\nabla^t(g)] = \int_M \operatorname{Tr}([\nabla, \nabla^t](g))$$

We obtain that the real moment map $\mu : \mathcal{A} \longrightarrow \mathfrak{P}_{\mathbb{C}}$ takes ∇ and puts it to $\frac{[\nabla, \nabla^t]}{\omega}$. ■