

# **Trihyperkähler reduction**

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**January 24, 2012,**

**IPMU, Tokyo**

## **Plan**

1. Symplectic reduction and GIT
2. Hyperkähler reduction
3. Quivers and ADHM construction
4. Trihyperkähler reduction and its applications

## Cartan's formula and symplectomorphisms

We denote the Lie derivative along a vector field as  $\text{Lie}_x : \Lambda^i M \longrightarrow \Lambda^i M$ , and contraction with a vector field by  $i_x : \Lambda^i M \longrightarrow \Lambda^{i-1} M$ .

**Cartan's formula:**  $d \circ i_x + i_x \circ d = \text{Lie}_x$ .

**REMARK:** Let  $(M, \omega)$  be a symplectic manifold,  $G$  a Lie group acting on  $M$  by symplectomorphisms, and  $\mathfrak{g}$  its Lie algebra. For any  $g \in \mathfrak{g}$ , denote by  $\rho_g$  the corresponding vector field. Then  $\text{Lie}_{\rho_g} \omega = 0$ , giving  $d(i_{\rho_g}(\omega)) = 0$ . **We obtain that  $i_{\rho_g}(\omega)$  is closed, for any  $g \in \mathfrak{g}$ .**

**DEFINITION:** **A Hamiltonian** of  $g \in \mathfrak{g}$  is a function  $h$  on  $M$  such that  $dh = i_{\rho_g}(\omega)$ .

## Moment maps

**DEFINITION:**  $(M, \omega)$  be a symplectic manifold,  $G$  a Lie group acting on  $M$  by symplectomorphisms. **A moment map**  $\mu$  of this action is a linear map  $\mathfrak{g} \rightarrow C^\infty M$  associating to each  $g \in \mathfrak{g}$  its Hamiltonian.

**REMARK:** It is more convenient to consider  $\mu$  as an element of  $\mathfrak{g}^* \otimes_{\mathbb{R}} C^\infty M$ , or (and this is most standard) **as a function with values in  $\mathfrak{g}^*$** .

**REMARK:** Moment map **always exists** if  $M$  is simply connected.

**DEFINITION:** A moment map  $M \rightarrow \mathfrak{g}^*$  is called **equivariant** if it is equivariant with respect to the coadjoint action of  $G$  on  $\mathfrak{g}^*$ .

**REMARK:**  $M \xrightarrow{\mu} \mathfrak{g}^*$  is a moment map iff for all  $g \in \mathfrak{g}$ ,  $\langle d\mu, g \rangle = i_{\rho_g}(\omega)$ . Therefore, **a moment map is defined up to a constant  $\mathfrak{g}^*$ -valued function**. An equivariant moment map is defined up to **a constant  $\mathfrak{g}^*$ -valued function which is  $G$ -invariant**.

**DEFINITION:** A  $G$ -invariant  $c \in \mathfrak{g}^*$  is called **central**.

**CLAIM:** **An equivariant moment map exists whenever  $H^1(G, \mathfrak{g}^*) = 0$** . In particular, when  $G$  is reductive and  $M$  is simply connected, an equivariant moment map exists.

## Symplectic reduction and GIT

**DEFINITION:** (Weinstein-Marsden)  $(M, \omega)$  be a symplectic manifold,  $G$  a compact Lie group acting on  $M$  by symplectomorphisms,  $M \xrightarrow{\mu} \mathfrak{g}^*$  an equivariant moment map, and  $c \in \mathfrak{g}^*$  a central element. The quotient  $\mu^{-1}(c)/G$  is called **symplectic reduction** of  $M$ , denoted by  $M//G$ .

**CLAIM:** The symplectic quotient  $M//G$  is a symplectic manifold of dimension  $\dim M - 2 \dim G$ .

**THEOREM:** Let  $(M, I, \omega)$  be a Kähler manifold,  $G_{\mathbb{C}}$  a complex reductive Lie group acting on  $M$  by holomorphic automorphisms, and  $G$  is compact form acting isometrically. **Then  $M//G$  is a Kähler orbifold.**

**REMARK:** In such a situation,  $M//G$  is called **the Kähler quotient**, or **GIT quotient**. The choice of a central element  $c \in \mathfrak{g}^*$  is known as a choice of **stability data**.

**REMARK:** **The points of  $M//G$  are in bijective correspondence with the orbits of  $G_{\mathbb{C}}$  which intersect  $\mu^{-1}(c)$ .** Such orbits are called **polystable**, and the intersection of a  $G_{\mathbb{C}}$ -orbit with  $\mu^{-1}(c)$  is a  $G$ -orbit.

## Hyperkähler manifolds

**DEFINITION:** A **hyperkähler structure** on a manifold  $M$  is a Riemannian structure  $g$  and a triple of complex structures  $I, J, K$ , satisfying quaternionic relations  $I \circ J = -J \circ I = K$ , such that  $g$  is Kähler for  $I, J, K$ .

**REMARK:** A hyperkähler manifold **has three symplectic forms**  
 $\omega_I := g(I\cdot, \cdot)$ ,  $\omega_J := g(J\cdot, \cdot)$ ,  $\omega_K := g(K\cdot, \cdot)$ .

**REMARK:** This is equivalent to  $\nabla I = \nabla J = \nabla K = 0$ : the parallel translation along the connection preserves  $I, J, K$ .

**REMARK:**

The form  $\Omega := \omega_J + \sqrt{-1} \omega_K$  is **holomorphic and symplectic** on  $(M, I)$ .

## Hyperkähler reduction

**DEFINITION:** Let  $G$  be a compact Lie group,  $\rho$  its action on a hyperkähler manifold  $M$  by hyperkähler isometries, and  $\mathfrak{g}^*$  a dual space to its Lie algebra. **A hyperkähler moment map** is a  $G$ -equivariant smooth map  $\mu : M \rightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$  such that  $\langle \mu_i(v), g \rangle = \omega_i(v, d\rho(g))$ , for every  $v \in TM$ ,  $g \in \mathfrak{g}$  and  $i = 1, 2, 3$ , where  $\omega_i$  is one three Kähler forms associated with the hyperkähler structure.

**DEFINITION:** Let  $\xi_1, \xi_2, \xi_3$  be three  $G$ -invariant vectors in  $\mathfrak{g}^*$ . The quotient manifold  $M // G := \mu^{-1}(\xi_1, \xi_2, \xi_3) / G$  is called **the hyperkähler quotient** of  $M$ .

**THEOREM:** (Hitchin, Karlhede, Lindström, Roček)

**The quotient  $M // G$  is hyperkaehler.**

## Holomorphic moment map

Let  $\Omega := \omega_J + \sqrt{-1}\omega_K$ . This is a holomorphic symplectic (2,0)-form on  $(M, I)$ .

**The proof of HKLR theorem. Step 1:** Let  $\mu_J, \mu_K$  be the moment map associated with  $\omega_J, \omega_K$ , and  $\mu_{\mathbb{C}} := \mu_J + \sqrt{-1}\mu_K$ . Then  $\langle d\mu_{\mathbb{C}}, g \rangle = i_{\rho g}(\Omega)$ . Therefore,  $d\mu_{\mathbb{C}} \in \Lambda^{1,0}(M, I) \otimes \mathfrak{g}^*$ .

**Step 2:** This implies that the map  $\mu_{\mathbb{C}}$  is holomorphic. It is called **the holomorphic moment map**.

**Step 3:** By definition,  $M // G = \mu_{\mathbb{C}}^{-1}(c) // G$ , where  $c \in \mathfrak{g}^* \otimes_{\mathbb{R}} \mathbb{C}$  is a central element. **This is a Kähler manifold**, because it is a Kähler quotient of a Kähler manifold.

**Step 4:** We obtain 3 complex structures  $I, J, K$  on the hyperkähler quotient  $M // G$ . **They are compatible in the usual way** (an easy exercise). ■



## Quiver representations

**DEFINITION:** A **quiver** is an oriented graph. A **quiver representation** is a diagram of complex Hermitian vector spaces and arrows associated with a quiver:

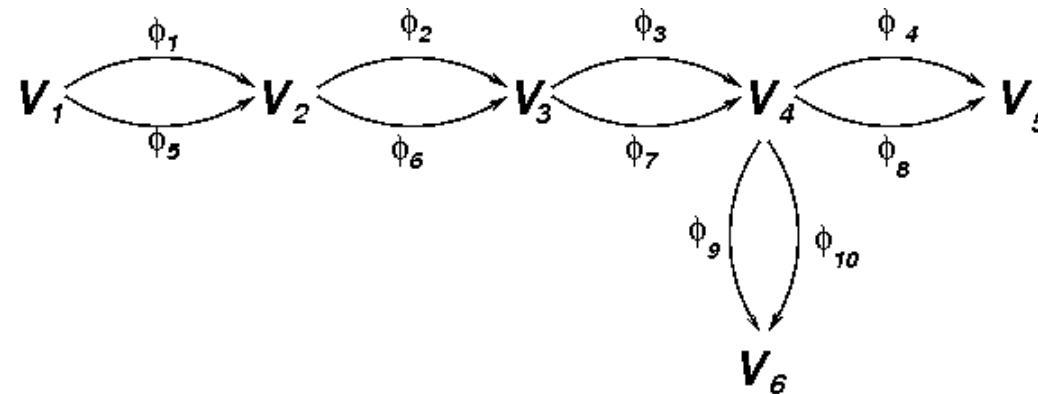
$$\begin{array}{ccccccccc} \mathbf{V}_1 & \xrightarrow{\phi_1} & \mathbf{V}_2 & \xrightarrow{\phi_2} & \mathbf{V}_3 & \xrightarrow{\phi_3} & \mathbf{V}_4 & \xrightarrow{\phi_4} & \mathbf{V}_5 \\ & & & & & & \downarrow \phi_9 & & \\ & & & & & & \mathbf{V}_6 & & \end{array}$$

Here,  $V_i$  are vector spaces, and  $\varphi_i$  linear maps.

**REMARK:** If one fixes the spaces  $V_i$ , the space of quiver representations is a Hermitian vector space.

## Quiver varieties

Starting from a single graph, one can double it up, as follows, obtaining a Nakajima double quiver.



A Nakajima quiver for the Dynkin diagram  $D_5$ .

**CLAIM:** The space  $M$  of representations of a Nakajima's double quiver is a quaternionic vector space, and the group  $G := U(V_1) \times U(V_2) \times \dots \times U(V_n)$  acts on  $M$  preserving the quaternionic structure.

**DEFINITION:** A **Nakajima quiver variety** is a quotient  $M // G$ .

## Hyperkähler manifolds as quiver varieties

Many non-compact hyperkähler manifolds are obtained as quiver varieties.

**EXAMPLE:** A 4-dimensional ALE (**asymptotically locally Euclidean**) space obtained as a resolution of **a du Val singularity**, that is, a quotient  $\mathbb{C}^2/G$ , where  $G \subset SU(2)$  is a finite group.

**REMARK:** Since finite subgroups of  $SU(2)$  are classified by the Dynkin diagrams of type A,D,E, these ALE quotients are called **ALE spaces of A-D-E type**.

**EXAMPLE:** The moduli asymptotically flat Hermitian Yang-Mills connections on ALE spaces of A-D-E type.

**DEFINITION:** **An instanton** on  $\mathbb{C}P^2$  is a stable bundle  $B$  with  $c_1(B) = 0$ . **A framed instanton** is an instanton equipped with a trivialization  $B|_C$  for a line  $C \subset \mathbb{C}P^2$ .

**THEOREM:** (Nahm, Atiyah, Hitchin) The space  $\mathcal{M}_{r,c}$  of framed instantons on  $\mathbb{C}P^2$  is **hyperkähler**.

**This theorem is proved using quivers.**

## ADHM construction

**DEFINITION:** Let  $V$  and  $W$  be complex vector spaces, with dimensions  $c$  and  $r$ , respectively. The **ADHM data** is maps

$$A, B \in \text{End}(V), I \in \text{Hom}(W, V), J \in \text{Hom}(V, W).$$

We say that ADHM data is

**stable**,

if there is no subspace  $S \subsetneq V$  such that  $A(S), B(S) \subset S$  and  $I(W) \subset S$ ;

**costable**,

if there is no nontrivial subspace  $S \subset V$  such that  $A(S), B(S) \subset S$  and  $S \subset \ker J$ ;

**regular**,

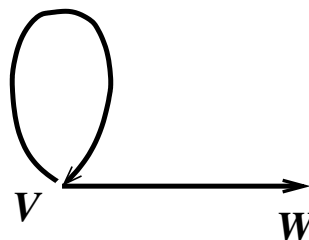
if it is both stable and costable.

**The ADHM equation** is  $[A, B] + IJ = 0$ .

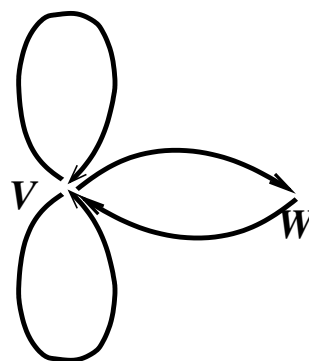
**THEOREM:** (Atiyah, Drinfeld, Hitchin, Manin) Framed rank  $r$ , charge  $c$  instantons on  $\mathbb{C}P^2$  are in bijective correspondence with the set of equivalence classes of regular ADHM solutions. In other words, **the moduli of instantons on  $\mathbb{C}P^2$  is identified with moduli of the corresponding quiver representation.**

## ADHM spaces as quiver varieties

Consider the quiver



The ADHM data is the set  $Q$  of representations of the corresponding double quiver



The corresponding **holomorphic moment map** is the **ADHM equation**  $A, B, I, J \rightarrow [A, B] + IJ$  with values in  $\text{End}(V)$ .

**The set of equivalence classes of ADHM solutions is  $Q // U(V)$ .**

## Twistor space

**DEFINITION: Induced complex structures** on a hyperkähler manifold are complex structures of form  $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$

**They are usually non-algebraic.** Indeed, if  $M$  is compact, for generic  $a, b, c$ ,  $(M, L)$  has no divisors (Fujiki).

**DEFINITION:** A **twistor space**  $\text{Tw}(M)$  of a hyperkähler manifold is a **complex manifold obtained by gluing these complex structures into a holomorphic family over  $\mathbb{C}P^1$ .** More formally:

Let  $\text{Tw}(M) := M \times S^2$ . Consider the complex structure  $I_m : T_m M \rightarrow T_m M$  on  $M$  induced by  $J \in S^2 \subset \mathbb{H}$ . Let  $I_J$  denote the complex structure on  $S^2 = \mathbb{C}P^1$ .

The operator  $I_{\text{Tw}} = I_m \oplus I_J : T_x \text{Tw}(M) \rightarrow T_x \text{Tw}(M)$  satisfies  $I_{\text{Tw}}^2 = -\text{Id}$ . **It defines an almost complex structure on  $\text{Tw}(M)$ .** This almost complex structure is known to be integrable (Obata, Salamon)

**EXAMPLE:** If  $M = \mathbb{H}^n$ ,  $\text{Tw}(M) = \text{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \setminus \mathbb{C}P^{2n-1}$

**REMARK: For  $M$  compact,  $\text{Tw}(M)$  never admits a Kähler structure.**

**Rational curves on  $\text{Tw}(M)$ .**

**REMARK:** The twistor space **has many rational curves**.

**DEFINITION:** Denote by  $\text{Sec}(M)$  **the space of holomorphic sections** of the twistor fibration  $\text{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$ .

**DEFINITION:** For each point  $m \in M$ , one has **a horizontal section**  $C_m := \{m\} \times \mathbb{C}P^1$  of  $\pi$ . The space of horizontal sections is denoted  $\text{Sec}_{hor}(M) \subset \text{Sec}(M)$

**REMARK:** The space of horizontal sections of  $\pi$  is identified with  $M$ . The normal bundle  $NC_m = \mathcal{O}(1)^{\dim M}$ . Therefore, **some neighbourhood of  $\text{Sec}_{hor}(M) \subset \text{Sec}(M)$  is a smooth manifold of dimension  $2 \dim M$ .**

**DEFINITION:** A twistor section  $C \subset \text{Tw}(M)$  is called **regular**, if  $NC = \mathcal{O}(1)^{\dim M}$ .

**CLAIM:** For any  $I \neq J \in \mathbb{C}P^1$ , consider the evaluation map  $\text{Sec}(M) \xrightarrow{E_{I,J}} (M, I) \times (M, J)$ ,  $s \longrightarrow s(I) \times s(J)$ . Then  **$E_{I,J}$  is an isomorphism around the set  $\text{Sec}_0(M)$  of regular sections.**

## Complexification of a hyperkähler manifold.

**REMARK:** Consider an anticomplex involution  $\text{Tw}(M) \xrightarrow{\iota} \text{Tw}(M)$  mapping  $(m, t)$  to  $(m, i(t))$ , where  $i : \mathbb{C}P^1 \rightarrow \mathbb{C}P^1$  is a central symmetry. Then  $\text{Sec}_{hor}(M) = M$  is a component of the fixed set of  $\iota$ .

**COROLLARY:**  $\text{Sec}(M)$  is a complexification of  $M$ .

**QUESTION:** What are geometric structures on  $\text{Sec}(M)$ ?

**Answer 1:** For compact  $M$ , a closure of  $\text{Sec}(M)$  is holomorphically convex (Stein if  $\dim M = 2$ ).

**Answer 2:** The space  $\text{Sec}_0(M)$  admits a holomorphic, torsion-free connection with holonomy  $Sp(n, \mathbb{C})$  acting on  $\mathbb{C}^{2n} \otimes \mathbb{C}^2$ .



## Mathematical instantons

**DEFINITION:** A **mathematical instanton** on  $\mathbb{C}P^3$  is a stable bundle  $B$  with  $c_1(B) = 0$  and  $H^1(B(-1)) = 0$ . A **framed instanton** is a mathematical instanton equipped with a trivialization of  $B|_\ell$  for some fixed line  $\ell = \mathbb{C}P^1 \subset \mathbb{C}P^3$ .

**DEFINITION:** An **instanton** on  $\mathbb{C}P^2$  is a stable bundle  $B$  with  $c_1(B) = 0$ . A **framed instanton** is an instanton equipped with a trivialization  $B|_x$  for some fixed point  $x \in \mathbb{C}P^2$ .

**THEOREM:** (Atiyah-Drinfeld-Hitchin-Manin) The space  $\mathcal{M}_{r,c}$  of framed instantons on  $\mathbb{C}P^2$  is **smooth, connected, hyperkähler**.

**THEOREM:** (Jardim–V.) The space  $\mathbb{M}_{r,c}$  of framed mathematical instantons on  $\mathbb{C}P^3$  **is naturally identified with the space of twistor sections  $\text{Sec}(\mathcal{M}_{r,c})$** .

**REMARK:** This correspondence is not surprising if one realizes that  $\text{Tw}(\mathbb{H}) = \mathbb{C}P^3 \setminus \mathbb{C}P^1$ .

## The space of instantons on $\mathbb{C}P^3$

**THEOREM:** (Jardim–V.) **The space  $\mathbb{M}_{r,c}$  is smooth.**

**REMARK:** To prove that  $\mathcal{M}_{r,c}$  is smooth, one could use hyperkähler reduction. To prove that  $\mathbb{M}_{r,c}$  is smooth, we develop **trihyperkähler reduction**, which is **a reduction defined on trisymplectic manifolds.**

We prove that  **$\mathbb{M}_{r,c}$  is a trihyperkähler quotient** of a vector space by a reductive group action, hence smooth.

## Trisymplectic manifolds

**DEFINITION:** Let  $\Omega$  be a 3-dimensional space of holomorphic symplectic 2-forms on a complex manifold. Suppose that

- $\Omega$  contains a non-degenerate 2-form
- For each non-zero degenerate  $\Omega \in \Omega$ , one has  $\text{rk } \Omega = \frac{1}{2} \dim V$ .

Then  $\Omega$  is called a **trisymplectic structure on  $M$** .

**REMARK:** The bundles  $\ker \Omega$  are involutive, because  $\Omega$  is closed.

**THEOREM:** (Jardim–V.) For any trisymplectic structure on  $M$ ,  $M$  is equipped with a unique holomorphic, torsion-free connection, preserving the forms  $\Omega_i$ . It is called **the Chern connection** of  $M$ .

**REMARK:** The Chern connection has holonomy in  $Sp(n, \mathbb{C})$  acting on  $\mathbb{C}^{2n} \otimes \mathbb{C}^2$ .

## Trisymplectic structure on $\text{Sec}_0(M)$

**EXAMPLE:** Consider a hyperkähler manifold  $M$ . Let  $I \in \mathbb{C}P^1$ , and  $ev_I : \text{Sec}_0(M) \rightarrow (M, I)$  be an evaluation map putting  $S \in \text{Sec}_0(M)$  to  $S(I)$ . Denote by  $\Omega_I$  the holomorphic symplectic form on  $(M, I)$ . **Then  $ev_I^* \Omega_I, I \in \mathbb{C}P^1$  generate a trisymplectic structure.**

**COROLLARY:**  $\text{Sec}_0(M)$  is equipped with a holomorphic, torsion-free connection with holonomy in  $Sp(n, \mathbb{C})$ .

## Trihyperkähler reduction

**DEFINITION:** A trisymplectic moment map  $\mu_{\mathbb{C}} : M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \Omega^*$  takes vectors  $\Omega \in \Omega, g \in \mathfrak{g} = \text{Lie}(G)$  and maps them to a holomorphic function  $f \in \mathcal{O}_M$ , such that  $df = \Omega \lrcorner g$ , where  $\Omega \lrcorner g$  denotes the contraction of  $\Omega$  and the vector field  $g$

**DEFINITION:** Let  $(M, \Omega, S_t)$  be a trisymplectic structure on a complex manifold  $M$ . Assume that  $M$  is equipped with an action of a compact Lie group  $G$  preserving  $\Omega$ , and an equivariant trisymplectic moment map

$$\mu_{\mathbb{C}} : M \longrightarrow \mathfrak{g}^* \otimes_{\mathbb{R}} \Omega^*.$$

Let  $\mu_{\mathbb{C}}^{-1}(0)$  be the corresponding **level set** of the moment map. Consider the action of the complex Lie group  $G_{\mathbb{C}}$  on  $\mu_{\mathbb{C}}^{-1}(c)$ . Assume that it is proper and free. Then the quotient  $\mu_{\mathbb{C}}^{-1}(c)/G_{\mathbb{C}}$  is a smooth manifold called **the trisymplectic quotient** of  $(M, \Omega, S_t)$ , denoted by  $M \text{ /// } G$ .

**THEOREM:** Suppose that the restriction of  $\Omega$  to  $\mathfrak{g} \subset TM$  is non-degenerate. **Then  $M \text{ /// } G$  is trisymplectic.**

## Mathematical instantons and the twistor correspondence

**REMARK:** Using the monad description of mathematical instantons, **we prove that that the map  $\text{Sec}_0(\mathcal{M}_{r,c}) \longrightarrow \mathbb{M}_{r,c}$  to the space of mathematical instantons is an isomorphism** (Frenkel-Jardim, Jardim-V.).

**REMARK:** The smoothness of the space  $\text{Sec}_0(\mathcal{M}_{r,c}) = \mathbb{M}_{r,c}$  **follows from the trihyperkähler reduction procedure:**

**THEOREM:** Let  $M$  be flat hyperkähler manifold, and  $G$  a compact Lie group acting on  $M$  by hyperkähler automorphisms. Suppose that the hyperkähler moment map exists, and the hyperkähler quotient  $M // G$  is smooth. **Then there exists an open embedding**

$$\text{Sec}_0(M) // G \xrightarrow{\Psi} \text{Sec}_0(M // G),$$

which is compatible with the trisymplectic structures on  $\text{Sec}_0(M) // G$  and  $\text{Sec}_0(M // G)$ .

**THEOREM:** If  $M$  is the space of quiver representations which gives  $M // G = \mathcal{M}_{2,c}$ ,  **$\Psi$  gives an isomorphism  $\text{Sec}_0(M) // G = \text{Sec}_0(M // G)$ .**

## The 1-dimensional ADHM construction

**DEFINITION:** Let  $V$  and  $W$  be complex vector spaces, with dimensions  $c$  and  $r$ , respectively. The **1-dimensional ADHM data** is maps

$$A_k, B_k \in \text{End}(V), I_k \in \text{Hom}(W, V), J_k \in \text{Hom}(V, W), (k = 0, 1)$$

Choose homogeneous coordinates  $[z_0 : z_1]$  on  $\mathbb{C}P^1$  and define

$$\tilde{A} := A_0 \otimes z_0 + A_1 \otimes z_1 \quad \text{and} \quad \tilde{B} := B_0 \otimes z_0 + B_1 \otimes z_1.$$

We say that 1-dimensional ADHM data is

**globally regular** if  $(\tilde{A}_p, \tilde{B}_p, \tilde{I}_p, \tilde{J}_p)$  is regular for every  $p \in \mathbb{C}P^1$ . The **1-dimensional ADHM equation** is  $[\tilde{A}_p, \tilde{B}_p] + \tilde{I}_p \tilde{J}_p = 0$ , for all  $p \in \mathbb{C}P^1$

**THEOREM:** (Marcos Jardim, Igor Frenkel) Let  $C_1(r, c)$  denote the set of globally regular solutions of the 1-dimensional ADHM equation. Then **there exists a 1-1 correspondence between equivalence classes of globally regular solutions of the 1-dimensional ADHM equations and isomorphism classes of rank  $r$  instanton bundles** on  $\mathbb{C}P^3$  framed at a fixed line  $\ell$ , where  $\dim W = \text{rk}(E)$  and  $\dim V = c_2(E)$ .

## 1-dimensional ADHM construction and the trisymplectic moment map

**THEOREM:** (Jardim, V.) Consider the natural (flat) trisymplectic structure on the space  $U_{r,c}$  of 1-dimensional ADHM data, and let  $\mu : U_{r,c} \rightarrow H^0(\mathcal{O}_{\mathbb{C}P^1}(1) \otimes \text{End}(V))$  be a map associating to  $C \in U_{r,c}$  and  $p \in \mathbb{C}P^1$  the vector  $[\tilde{A}_p, \tilde{B}_p] + \tilde{I}_p \tilde{J}_p \in \mathcal{O}_{\mathbb{C}P^1}(2) \otimes \text{End}(V)$ . **Then  $\mu$  is a trisymplectic moment map. This identifies the set of equivalence classes of solutions of the 1-dimensional ADHM equation with the trihyperkähler quotient  $U_{r,c} // // U(V)$ .**