Rigid currents on hyperkähler manifolds

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Hyperkähler manifolds

DEFINITION: A **hypercomplex manifold** is a manifold M equipped with three complex structure operators I, J, K, satisfying **quaternionic relations**

$$IJ = -JI = K, \quad I^2 = J^2 = K^2 = -\operatorname{Id}_{TM}$$

DEFINITION: A **hyperkähler manifold** is a hypercomplex manifold equipped with a metric g which is Kähler with respect to I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **maximal holonomy**, or **IHS** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be of maximal holonomy.

Gromov-Hausdorff metrics

DEFINITION: Let $X \subset M$ be a subset of a metric space, and $y \in M$ a point. **Distance from a y to X** is $\inf_{x\in X} d(x,y)$. **Hausdorff distance** $d_H(X,Y)$ between to subsets $X,Y \subset M$ of a metric space is

$$\max(\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)).$$

Gromov-Hausdorff distance between complete metric spaces X,Y of diameter $\leqslant d$ is an infimum of $d_H(\varphi(X),\psi(Y))$ taken over all isometric embeddings $\varphi:\ X \longrightarrow Z$, $\psi:\ Y \longrightarrow Z$ to a third metric space.

REMARK: It is not hard to see that **a converging sequence of Riemannian metrics converges in Gromov-Hausdorff topology.**

Gromov's compactness theorem

DEFINITION: A subset $X \subset M$ is called **precompact** if its closure in M is compact.

DEFINITION: We say that **Ricci curvature of a Riemannian manifold (M, g) is bounded from below by c** if the symmetric form $\operatorname{Ric}_g - cg \in \operatorname{Sym}^2 T * M$ is positive definite.

THEOREM: (Gromov's compactness theorem)
Let W_d be the Gromov's space of all metric spaces of diameter d, and $X_{c,d} \subset W_d$ the space of all Riemannian manifolds with Ricci curvature bounded from below by c. **Then $X_{c,d}$ is precompact.**

QUESTION: Let Hyp_d be the space of all hyperkähler metrics of diameter d considered as a subset in W_d. **What is the shape of Hyp_d and its closure?**

Space of hyperkähler metrics and its closure

THEOREM: Let (M,I) be a hyperkähler manifold with ergodic complex structure, and $\overline{V}_I$ the Gromov-Hausdorff closure of the space V_I of all hyperkähler metrics on M. **Then $\overline{V}_I$ contains the space Hyp of all hyperkähler metrics on M obtained by deformation from V_I.**

REMARK: $\dim V_I = b_2 - 2$, and $\dim \operatorname{Hyp} = 3b_2 - 8$: much bigger!

Related question: Consider hyperkähler forms ω_I as currents on (M,I), and let $\overline{\operatorname{Hyp}}_{cur}$ be its closure in the space of currents. **Is it related by $\overline{\operatorname{Hyp}}_d$?**

THEOREM: Let $[\omega_0] \in H^{1,1}(M,I)$ be a nef class such that $[\omega_0]^\perp \cap H^2(M,\mathbb{Q}) = 0$. Then for any sequence of Kähler forms ω_i on (M,I) such that with $[\omega_i]$ converges to $[\omega_0] \in H^{1,1}(M)$, **the sequence ω_i converges to a unique positive current.**

Currents and generalized functions

DEFINITION: Let F be a Hermitian bundle with connection ∇, on a Riemannian manifold M with Levi-Civita connection, and

$$\|f\|_{C^k} := \sup_{x\in M} \left(|f| + |\nabla f| + ... + |\nabla^k f|\right)$$

the corresponding **C^k-norm** defined on smooth sections with compact support. **The C^k-topology is independent from the choice of connection and metrics.**

DEFINITION: A generalized function is a functional on top forms with compact support, which is continuous in one of C^i-topologies.

DEFINITION: A k-current is a functional on $(\dim M - k)$-forms with compact support, which is continuous in one of C^i-topologies.

REMARK: Currents are forms with coefficients in generalized functions.

Currents on complex manifolds

DEFINITION: The space of currents is equipped with **weak topology** (a sequence of currents converges if it converges on all forms with compact support). The space of currents with this topology is a **Montel space** (barrelled, locally convex, all bounded subsets are precompact). Montel spaces are **reflexive** (the map to its double dual with strong topology is an isomorphism).

CLAIM: De Rham differential is continuous on currents, and the Poincare lemma holds. Hence, **the cohomology of currents are the same as cohomology of smooth forms.**

DEFINITION: On an complex manifold, **(p,q)-currents** are (p,q)-forms with coefficients in generalized functions

REMARK: In the literature, this is sometimes called $(n-p,n-q)$-currents.

CLAIM: The Dolbeault lemma holds on (p,q)-currents, and **the $\overline{\partial}$-cohomology are the same as for forms.**

Positive currents and measures

DEFINITION: We inter[ret sections of $\Lambda^{n-1,n-1}(M)$ as Vol_M-valued pseudo-Hermitian forms on T^*M. A form $\eta \in \Lambda^{n-1,n-1}(M)$ is **positive** if for any $x \in T^*M$, one has $\eta(x, Ix) \geqslant 0$.

DEFINITION: A **positive $(1,1)$-current** is a current taking non-negative values on positive compactly supported $(n-1, n-1)$-forms.

DEFINITION: A **positive generalized function** is a generalized function taking non-negative values on all positive volume forms.

REMARK: Positive generalized functions are C^0-continuous. A positive generalized function multiplied by a positive volume form **gives a measure on a manifold,** and **all measures are obtained this way.**

DEFINITION: A **mass** of a positive $(1,1)$-current η on a Hermitian n-manifold (M,ω) is a measure $\eta \wedge \omega^{n-1}$. **It is non-negative, and positive if $\eta \neq 0$.**

Positive currents: compactness theorem

Theorem: The space of positive $(1,1)$-currents with bounded mass is (weakly) compact.

Proof: Follows from precompactness of the space of bounded measures in weak-*-topology. ■

Rigid currents

DEFINITION: **A nef class** is a limit of Kähler $(1,1)$-classes in $H^{1,1}(M)$.

DEFINITION: **A nef current** is a limit of positive, closed $(1,1)$-forms in the space of currents.

REMARK: **All nef classes can be represented by nef currents** (by compactnes).

DEFINITION: A nef class is called **rigid** if it has a unique positive, closed representative in the space of currents.

THEOREM: **(Sibony, V.)**
Let η be a nef class on a hyperkähler manifold M, $\dim_{\mathbb{C}} M = 2n$. Assume that $\int_M \eta^{2n} = 0$, and $\eta^\perp \cap H^2(M, \mathbb{Q}) = 0$. **Then the corresponding nef current is rigid.**

Hyperbolic automorphisms

DEFINITION: An automorphism of a hyperkähler manifold is called **hyperbolic** if it acts on $H^{1,1}(M)$ with eigenvalue $\alpha \in \mathbb{R}$, where $\alpha > 1$.

REMARK: The corresponding eigenspace is 1-dimensional.

THEOREM: (Amerik, V.) Every hyperkähler manifold with $b_2 > 4$ **has a deformation which admits a hyperbolic automorphism.**

DEFINITION: A class $v \in H^{1,1}(M)$ on a hyperkähler manifold is called **hyperbolic** if M admits a hyperbolic automorphism T, and v is its eigenvector with eigenvalue $\alpha > 1$.

Rigidiy of hyperbolic currents

THEOREM: (Cantat, Dinh-Sibony)
Let $v \in H^{1,1}(M)$ be a hyperbolic class. **Then v is nef and rigid.**

Proof. Step 1: The class v is nef. Indeed, $\lim \frac{T^n\omega}{\alpha^n} = v$ for any $\omega \notin V$, where $V \subset H^{1,1}(M, \mathbb{R})$ is a subspace of positive codimension. Taking ω Kähler (the Kähler cone is open, hence we can assume that $\omega \notin V$), we obtain that v is nef.

Step 2: It remains to prove **uniqueness of the positive representative η of v.** Suppose that there are two positive representatives η_1, η_2, with $\eta_1 - \eta_2 = dd^c\psi$ by dd^c-lemma. The generalized function ψ is determined by this equation uniquely up to a constant, because **a dd^c-closed generalized function is constant.** Then $T^*\psi = \alpha\psi + C$, where C is a real constant. Since the canonical bundle K_M is trivial, M has a T-invariant volume form Vol. Consider the pushforward ψ_*Vol as a measure on $\mathbb{R}$. Then ψ_*Vol is mapped to itself by $x \longrightarrow \alpha x + C$. This is impossible, however, because such a measure must be atomic, and ψ is non-constant. ■

Teichmüller spaces

DEFINITION: Let M be a smooth manifold. **A complex structure** on M is an endomorphism $I \in \operatorname{End} TM$, $I^2 = -\operatorname{Id}_{TM}$ such that the eigenspace bundles of I are **involutive**, that is, satisfy satisfy $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$. Denote by Comp the space of such tensors equipped with a topology of convergence of all derivatives.

DEFINITION: Let M be a compact complex manifold, $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group $\operatorname{Diff}(M)$ (**the group of isotopies**). Let $\operatorname{Teich} := \operatorname{Comp} / \operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

DEFINITION: The group $\Gamma := \operatorname{Diff}(M)/\operatorname{Diff}_0(M)$ is called **the mapping class group**; it acts on Teich in a natural way, and the quotient set is the set of all complex structures on M up to equivalence.

REMARK: Working with hyperkähler manifolds, **we shall restrict ourself to an open subset of** Teich **consisting of all complex structures compatible with a hyperkähler structure.**

Computation of the mapping class group

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$,** and $c > 0$ a rational number.

DEFINITION: The form q is called **Bogomolov-Beauville-Fujiki form**. It has signature $(3, b_2 - 3)$.

THEOREM: (V., 1996, 2009) Let M be a maximal holonomy, compact hyperkähler manifold, and $\Gamma_0 = \operatorname{Aut}(H^*(M,\mathbb{Z}), p_1, ..., p_n)$. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}), q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}), q)$ **has finite kernel.**
(iii) The tautological map $\Gamma \longrightarrow \Gamma_0$ **has finite kernel and its image has finite index,** where Γ is a mapping class group.

The period map

REMARK: To simplify the language, we redefine Teich and Comp for hyperkähler manifolds, admitting only complex structures of Kähler type. Since the Hodge numbers are constant in families of Kähler manifolds, **for any $J \in$ Teich, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.**

Definition: Let Per : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map Per : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

REMARK: Per **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}\mathrm{er} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \ | \ \ q(l, l) = 0, q(l, \bar{l}) > 0\}.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$. Indeed, the group $SO(H^2(M, \mathbb{R}), q) = SO(b_2 - 3, 3)$ acts transitively on $\mathbb{P}\mathrm{er}$, and $SO(2) \times SO(b_2 - 3, 1)$ is a stabilizer of a point.

Birational Teichmüller moduli space

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (Huybrechts, 2001) Two points $I, I' \in$ Teich **are non-separable if and only if there exists a bimeromorphism $(M, I) \longrightarrow (M, I')$ which is non-singular in codimension 2 and acts as identity on $H^2(M)$.**

REMARK: This is possible only if (M, I) and (M, I') contain a rational curve. **General hyperkähler manifold has no curves;** ones which have belong to a countable union of divisors in Teich.

DEFINITION: The space $\operatorname{Teich}_b :=$ Teich $/ \sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ is an isomorphism, for each connected component of Teich_b.

Uniqueness of positive representative

THEOREM: Let (M,I) be a hyperkähler manifold with Pic(M) non-maximal. Assume that a deformation of M admits a hyperbolic automorphism. Consider a nef class $\mu \in H^{1,1}(M)$, with $\mu^\perp \cap H^2(M,\mathbb{Q}) = 0$. **Then μ is rigid.**

Proof. Step 1: Let S_x be the set of all positive currents representing a cohomology class $x \in H^{1,1}(M)$. This is a compact set. By dd^c-lemma, for all $\eta, \eta' \in S_x$, one has $\eta - \eta' = dd^c f$ for some generalized function f. **The function f is unique up to a constant;** we chose the constant in such a way that $\int_M f \operatorname{Vol}_M = 0$. Let $\delta(x) := \sup_{\eta,\eta'} \int_M |f| \operatorname{Vol}_M$. This number is bounded by compactness of the space of positive currents, and $\delta(x) = 0$ **if and only if x is a rigid current.**

Step 2: Let Teich$_p$ be the Teichmüller space of pairs (M,I,η) where $\eta \in H^{1,1}(M,I)$ is a nef class with $\int_M \eta^{2n} = 0$. The mapping class group Γ acts on each component of Teich$_\eta$ with dense orbits, and the function $(I,\eta) \longrightarrow \delta(\eta)$ is upper semi-continuous. Therefore, **it reaches its minimum on any dense orbit of Γ.**

Step 3: The orbit of (M,I,μ) is dense in Teich$_p$ by Ratner theorem. On the other hand, $\delta(v) = 0$ for any hyperbolic class $v \in H^{1,1}(M)$. Then $\delta(\mu) = 0$. ■