Hypercomplex manifolds of quaternionic dimension 2 and HKT-structures

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Hypercomplex manifolds

DEFINITION: Let M be a smooth manifold equipped with endomorphisms $I, J, K : TM \longrightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}$. Suppose that I, J, K are integrable almost complex structures. Then (M, I, J, K) is called **a hypercomplex manifold**.

THEOREM: (Obata, 1955) On any hypercomplex manifold **there exists a unique torsion-free connection ∇ such that $\nabla I = \nabla J = \nabla K$.**

DEFINITION: Such a connection is called **the Obata connection**.

REMARK: The holonomy of Obata connection lies in $GL(n,\mathbb{H})$.

REMARK: A torsion-free connection ∇ on M with $\mathcal{H}\mathrm{ol}(\nabla) \subset GL(n,\mathbb{H})$ **defines a hypercomplex structure on M.**

Examples of hypercomplex manifolds

EXAMPLE: A **Hopf surface** $M = \mathbb{H}\backslash 0/\mathbb{Z} \cong S^1 \times S^3$. The holonomy of Obata connection $\mathcal{H}\mathrm{ol}(M) = 0$.

EXAMPLE: **Compact holomorphically symplectic manifolds are hyperkähler (by Calabi-Yau theorem)**, hence hypercomplex. Here $\mathcal{H}\mathrm{ol}(M) \subset Sp(n)$ **(this is equivalent to being hyperkähler).**

PROPOSITION: A compact hypercomplex manifold (M, I, J, K) with (M, I) of Kähler type also admits a hyperkähler structure.

REMARK: In dimension 1, compact hypercomplex manifolds are classified (C. P. Boyer, 1988). This is the complete list: **torus, K3 surface, Hopf surface**.

Examples of hypercomplex manifolds (2)

EXAMPLE: The Lie groups

$$\begin{array}{llll} SU(2l+1), & T^1 \times SU(2l), & T^l \times SO(2l+1), & \\ T^{2l} \times SO(4l), & T^l \times Sp(l), & T^2 \times E_6, & \\ T^7 \times E^7, & T^8 \times E^8, & T^4 \times F_4, & T^2 \times G_2. \end{array}$$

Some other homogeneous spaces (D. Joyce and physicists Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen). **Holonomy unknown (but likely $GL(n,\mathbb{H})$).**

THEOREM: (A. Soldatenkov)
Holonomy of Obata connection on $SU(3)$ is $GL(2,\mathbb{H})$.

EXAMPLE: Many **nilmanifolds** (quotients of a nilpotent Lie group by a cocompact lattice) admit hypercomplex structures. **In this case $\mathcal{H}\mathrm{ol}(M) \subset SL(n,\mathbb{H})$.**

Quaternionic Hermitian structures

DEFINITION: Let (M,I,J,K) be a hypercomplex manifold, and g a Riemannian metric. We say that g is **quaternionic Hermitian** if I,J,K are orthogonal with respect to g.

CLAIM: Quaternionic Hermitian metrics always exist.

Proof: Take any Riemannian metric g and **consider its average** $\operatorname{Av}_{SU(2)} g$ with respect to $SU(2) \subset \mathbb{H}^*$. ∎

Given a quaternionic Hermitian metric g on (M,I,J,K), consider its Hermitian forms

$$\omega_I(\cdot,\cdot) = g(\cdot, I\cdot), \omega_J, \omega_K$$

(real, but *not closed*). Then $\Omega = \omega_J + \sqrt{-1}\omega_K$ is of Hodge type (2,0) with respect to I.

If $d\Omega = 0$, (M,I,J,K,g) **is hyperkähler** (this is one of the definitions).

Consider a weaker condition:

$$\partial\Omega = 0, \quad \partial : \Lambda^{2,0}(M,I) \longrightarrow \Lambda^{3,0}(M,I)$$

HKT structures

DEFINITION: (Howe, Papadopoulos, 1998)
Let (M,I,J,K) be a hypercomplex manifold, g a quaternionic Hermitian metric, and $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ the corresponding $(2,0)$-form. We say that g is **HKT ("hyperkähler with torsion")** if $\partial\Omega = 0$..

HKT-metrics play in hypercomplex geometry the same role as Kähler metrics play in complex geometry.

1. **They admit a smooth potential** (locally). There is a notion of an "HKT-class" (similar to Kähler class) in a certain finite-dimensional coholology group. Two metrics in the same HKT-class differ by a potential, which is a function.

2. When (M,I) has trivial canonical bundle, **a version of Hodge theory is established** giving an $\mathfrak{sl}(2)$-action on holomorphic cohomology $H^*(M,\mathcal{O}_{(M,I)})$ and analogue of Hodge decomposition and dd^c-lemma.

3. **Not all compact hypercomplex manifolds are HKT** (Fino, Grantcharov).

Buchsdahl-Lamari theorem and its hypercomplex analogue

THEOREM: (Buchsdahl-Lamari)
Let M be a compact complex surface. Then M **is Kähler if and only if** $b_1(M)$ **is even.**

DEFINITION: A compact hypercomplex manifold with Obata holonomy in $SL(n,\mathbb{H}) \subset GL(n,\mathbb{H})$ is called $SL(n,\mathbb{H})$**-manifold.**

THEOREM: (Grantcharov-Lejmi-V.)
Let M be an $SL(2,\mathbb{H})$-manifold. Then M **is HKT if and only if** $h^1(\mathcal{O}_{M,I})$ **is even.**

REMARK: Using the Hodge decomposition on $H^*(\mathcal{O}_{M,I})$, one can show that $h^1(\mathcal{O}_{M,I})$ **is even for any** $SL(n,\mathbb{H})$**-manifold admitting an HKT structure.**

Plan of this talk:

1. Introduce Hodge theory on hypercomplex manifolds.

a. HKT-structures.

b. Canonical bundle.

c. Quaternionic Dolbeault complex.

d. Laplacians and cohomology.

2. Explain Harvey-Lawson and Lamari's ideas used in the proof of Buchsdahl-Lamari theorem.

3. Deduce the main result from an HKT-analogue of dd^c-lemma.

dd^c-lemma and its applications

LEMMA: Let (M,ω) be a compact Kähler manifold, and η d^c-closed, d-exact form. **Then $\eta \in \operatorname{im} dd^c$, where $d^c = IdI$.**

Proof. Step 1: Kodaira identities gives $\Delta = \{d, d^*\} = \{d^c, (d^c)^*\}$, where d^c is an anticommutator. Denote by $\Lambda = *L*$ the Hermitian conjugate of $L(\alpha) = \alpha\wedge\omega$. The Laplacian identity is deduced from $(d^c)^* = -[\Lambda, d]$ and $d^* = [\Lambda, d^c]$. Then, for any d, d^c-closed form α, this gives $\Delta(\alpha) = dd^c\Lambda\alpha$.

Step 2: Let G_Δ be the Green operator, equal to Δ^{-1} on orthogonal complement to $\ker\Delta$, and vanishing on $\ker\Delta$. Then G_Δ commutes with all operators commuting with Δ. Now, η is d-exact, hence orthogonal to $\ker\Delta$. This gives $\eta = G_\Delta\Delta\eta = G_\Delta dd^c\Lambda\eta = dd^c G_\Delta\Lambda\eta$ ■

Applications of dd^c-lemma:

1. Formality in rational homotopy.

2. Unobstructedness of deformations.

3. Existence of a metric on a holomorphic line bundle with prescribed curvature.

HKT potential

Defining Kähler metric via Kähler potentials: A Kähler metric on (M,I) is one which is locally given as

$$g(\cdot,\cdot) = \sqrt{-1}\,\partial\bar\partial\varphi(\cdot, I\cdot)$$

where φ is a function called **a Kähler potential**.

Defining HKT metric through HKT potentials: An HKT metric on (M,I) is one which is locally given as

$$g(\cdot,\cdot) = D(\varphi), \quad \text{where} \quad D(\varphi) := \mathrm{Av}_{SU(2)}(\sqrt{-1}\,\partial\bar\partial\varphi(\cdot, I\cdot))$$

and φ is a function called **an HKT potential**.

THEOREM: (Banos-Swann)
This definition is equivalent to the usual one.

DEFINITION: A function which is an HKT potential of some HKT metric is called **strictly $\mathbb{H}$-plurisubharmonic**, or $\mathbb{H}$-psh.

REMARK: For any $\mathbb{H}$-psh function φ, **φ is subharmonic** with respect to any quaternionic Hermitian metric. Therefore, **there are no globally defined $\mathbb{H}$-psh functions** on compact manifolds.

HKT-forms

DEFINITION: Let g be an HKT metric. The corresponding $(2,0)$-form $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ is called **an HKT-form**.

CLAIM: Consider the multiplicative action of J on $\Lambda^*(M)$. **Then J maps $\Lambda^{p,q}(M))$ to $\Lambda^{q,p}(M))$.**

Proof: I and J anticommute. ∎

DEFINITION: A (2,0)-form Ω on (M,I) is called **real** if $J(\Omega) = \overline{\Omega}$ and **strictly positive** if $\Omega(x, J(\overline{x})) > 0$ for each non-zero $x \in T_I^{1,0}(M)$.

CLAIM: Any HKT-form is strictly positive and real. Moreover, **any ∂-closed strictly positive real form $\Omega \in \Lambda_I^{2,0}(M)$ defines an HKT-metric** $g(x,y) := \Omega(x, J(\overline{y}))$.

Canonical bundle of a hypercomplex manifold.

0. Quaternionic Hermitian structure always exists.

1. **Complex dimension is even.**

2. **The canonical line bundle $\Lambda^{n,0}(M,I)$ of (M,I) is always trivial topologically.** Indeed, a non-degenerate section of canonical line bundle is provided by top power of a form Ω associated with some quaternionic Hermitian strucure. In particular, $c_1(M,I)=0$.

3. Canonical bundle **is non-trivial holomorphically** in many cases. However, **when M is a nilmanifold**, $\Lambda^{n,0}(M,I)$ is trivial, and holonomy of Obata connection lies in $SL(n,\mathbb{H})$ (Barberis-Dotti-V., 2007)

4. If $\mathcal{H}\text{ol}(M)$ lies in $SL(n,\mathbb{H})$, canonical bundle is trivial. The converse is true when M is compact and HKT (V., 2004): **an HKT manifold with holomorphically trivial canonical bunlde satisfies $\mathcal{H}\text{ol}(M)\subset SL(n,\mathbb{H})$.**

$SU(2)$-action on $\Lambda^*(M)$

The group $SU(2)$ of unitary quaternions acts on TM, because quaternion algebra acts. By multilinearity, this action is extended to $\Lambda^*(M)$.

1. The Hodge decomposition $\Lambda^*(M) = \bigoplus \Lambda^{p,q}(M)$ is recovered from this $SU(2)$-action. **"Hypercomplex analogue of the Hodge decomposition".**

2. $\langle \omega_I, \omega_J, \omega_K\rangle$ is an irreducible 3-dimensional representation of $SU(2)$, for any quaternionic Hermitian structure ("representation of weight 2").

WEIGHT of a representation.
We say that an irreducible $SU(2)$-representation W **has weight** i if $\dim W = i+1$. A representation is said to be **pure of weight** i if all its irreducible components have weight i. If all irreducible components of a representation W_1 have weight $\leqslant i$, we say that W_1 **is a representation of weight** $\leqslant i$. In a similar fashion one defines representations of weight $\geqslant i$.

Quaternionic Dolbeault algebra

The weight is multiplicative, in the following sense: a tensor product of representations of weights $\leqslant i$ and $\leqslant j$ has weight $\leqslant i+j$.

Clearly, $\Lambda^1(M)$ has weight 1. Therefore, **$\Lambda^i(M)$ has weight $\leqslant i$.**

Let $V^i \subset \Lambda^i(M)$ be the maximal $SU(2)$-invariant subspace of weight $< i$.

By multiplicativity, **$V^* = \bigoplus_i V^i$ is an ideal in $\Lambda^*(M)$**. We also have $V^i = \Lambda^i(M)$ for $i > 2n$. Also, $dV^i \subset V^{i+1}$, hence $V^* \subset \Lambda^*(M)$ is a differential ideal in $(\Lambda^i(M), d)$.

Denote by $(\Lambda^*_+(M), d_+)$ the quotient algebra $\Lambda^*(M)/V^*$.
We call it **the quaternionic Dolbeault algebra (qD-algebra)** of M.

A similar construction was given by Salamon in a more general situation.

The Hodge decomposition of quaternionic Dolbeault algebra.

The Hodge decomposition is induced from the $SU(2)$-action, hence it is **compatible with weights**: $\Lambda^i_+(M) = \bigoplus_{p+q=i} \Lambda^{p,q}_{+,I}(M)$.

Let $\sqrt{-1}\,\mathcal{I}$ be an element of the Lie algebra $\mathfrak{su}(2)\otimes\mathbb{C}$ acting as $\sqrt{-1}\,(p-q)$ on $\Lambda^{p,q}(M)$. This vector generates the Cartan algebra of $\mathfrak{su}(2)$. The $\mathfrak{su}(2)$-action induces an isomorphism of $\Lambda^{p,q}_{+,I}(M)$ for all $\{p,q \mid p+q=k,\ p,q\geqslant 0\}$. This gives

Theorem: $\Lambda^{p,q}_{+,I}(M) \cong \Lambda^{0,p+q}(M,I)$.

This isomorphism is provided by the $\mathfrak{su}(2)\otimes\mathbb{C}$-action.

Differentials in the qD-complex

We extend $J: \Lambda^1(M) \longrightarrow \Lambda^1(M)$ to $\Lambda^*(M)$ by multiplicativity. Since I and J anticommute on $\Lambda^1(M)$, **we have** $J(\Lambda^{p,q}(M,I)) = \Lambda^{q,p}(M,I)$.

Denote by $\partial_J: \Lambda^{p,0}(M,I) \longrightarrow \Lambda^{p,0}(M,I)$ **the operator** $J \circ \overline{\partial} \circ J$**,** where $\overline{\partial}: \Lambda^{0,p}(M,I) \longrightarrow \Lambda^{0,p}(M,I)$ is the standard Dolbeault differential. Then ∂, ∂_J anticommute. Moreover, **there exists a multiplicative isomorphism of bicomplexes**.

$$
\begin{tikzcd}
 & & \Lambda^0_+(M) \arrow[dl, "d_+^{1,0}"'] \arrow[dr, "d_+^{0,1}"] & & \\
 & \Lambda^{1,0}_+(M) \arrow[dl, "d_+^{1,0}"'] \arrow[dr, "d_+^{0,1}"] & & \Lambda^{0,1}_+(M) \arrow[dl, "d_+^{1,0}"'] \arrow[dr, "d_+^{0,1}"] & \\
\Lambda^{2,0}_+(M) & & \Lambda^{1,1}_+(M) & & \Lambda^{0,2}_+(M) \\
 & & \cdots & &
\end{tikzcd}
\quad \cong \quad
\begin{tikzcd}
 & & \Lambda^{0,0}_I(M) \arrow[dl, "\partial_J"'] \arrow[dr, "\partial"] & & \\
 & \Lambda^{1,0}_I(M) \arrow[dl, "\partial_J"'] \arrow[dr, "\partial"] & & \Lambda^{1,0}_I(M) \arrow[dl, "\partial_J"'] \arrow[dr, "\partial"] & \\
\Lambda^{2,0}_I(M) & & \Lambda^{2,0}_I(M) & & \Lambda^{2,0}_I(M) \\
 & & \cdots & &
\end{tikzcd}
$$

Potentials for HKT-metrics

A quaternionic Hermitian metric **can be recovered from the corresponding (2,0)-form:** $\omega_I(x,\overline{y}) = \frac{1}{2}\Omega(x,J(\overline{y}))$, where $x,y\in T^{1,0}(M)$. The HKT-structures uniquely correspond to $(2,0)$-forms which are

1. **Real:** $J(\Omega)=\overline{\Omega}$

2. **Closed:** $\partial\Omega = 0$.

2. **Positive:** $\Omega(x,J(\overline{x}))>0$, for any non-zero $x\in T^{1,0}(M)$

Locally, any HKT-metric is given by a potential: $\Omega = \partial\partial_J\varphi$ where φ is a smooth function.

Any convex, and any strictly plurisubharmonic function is a potential of some HKT-structure. Therefore, HKT-structures locally always exist (Grantcharov, Poon).

DEFINITION: Quaternionic Hessian of $f\in C^\infty M$ is a form $x,y\longrightarrow \partial\partial_J\varphi(x,J(\overline{x}))$. It is equal to the usual Hessian averaged with $SU(2)$. A function is **quaternionic plurisubharmonic** if its $\mathbb{H}$-Hessian is positive; equivalently, if $\partial\partial_J f$ is a positive $(2,0)$-form.

Hodge theory on HKT-manifolds with holonomy in $SL(n,\mathbb{H})$

DEFINITION: Let Φ be a non-degenerate, real, Obata-parallel section of $\Lambda^{n,0}(M,I)$. Then (M,J,J,K,Φ) is called **an $SL(n,\mathbb{H})$-manifold**.

DEFINITION: Let M be a compact HKT-manifold with holonomy in $SL(n,\mathbb{H})$, and $\Delta_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. Then $\ker \Delta_{\overline{\partial}}\big|_{\Lambda^{0,*}(M)} = H^*(M,\mathcal{O}_{(M,I)})$.

THEOREM: $\Delta_{\overline{\partial}}$ commutes with the multiplication by the HKT-form $\overline{\Omega}$, and with the operator $\eta \longrightarrow J(\overline{\eta})$. In particular, **there is a Lefschetz-like $\mathfrak{sl}(2)$-action on $H^*(M,\mathcal{O}_{(M,I)})$.**

REMARK: To simplify notation, it is more convenient to consider the ∂ and ∂_J-Laplacian, and to identify $H^*(M,\mathcal{O}_{(M,I)})$ with the cohomology of ∂.

THEOREM: The Laplacians Δ_∂ and Δ_{∂_J} are equal. In particular, $\eta \longrightarrow J(\overline{\eta})$ defines a complex structure on Δ_∂-harmonic $(k,0)$-forms for odd k, and real structure for odd k.

Theorem ("$\partial\partial_J$-lemma") Let η be a ∂_J-closed, ∂-exact form on an HKT $SL(n,\mathbb{H})$-manifold. **Then η is $\partial\partial_J$-exact: $\eta \in \operatorname{im} \partial\partial_J \eta$.**

Hahn-Banach separation theorem and its applications

THEOREM: Let V be a locally convex topological vector space, $W \subset V$ a closed subspace, and $A \subset V$ an open, convex subset, not intersecting A. **Then there exists a continuous linear functional $\xi \in V^*$ vanishing on W and positive on A.**

THEOREM: (Harvey, Lawson):
Let M be a compact complex non-Kähler manifold. Then there exists a positive $(n-1,n-1)$-current ξ which is a $(n-1,n-1)$-part of an exact current.

Idea of a proof: Hahn-Banach separation theorem is applied to the set A of strictly positive $(1,1)$-forms, and the set W of closed $(1,1)$-forms, obtaining a current $\xi \in D^{n-1,n-1}(M) = \Lambda^{1,1}(M)^*$ positive on A (that is, positive) and vanishing on W. The latter condition is equivalent to "$(n-1,n-1)$-part of an exact current". ■

Lamari's proof of Buchsdahl-Lamari theorem

THEOREM: (Buchsdahl-Lamari)
Let M be a compact complex surface. Then M **is Kähler if and only if** $b_1(M)$ **is even.**

Scheme of Lamari's proof:
Step 1: Evenness of $b_1(M)$ is equivalent to dd^c-lemma.

Step 2: Using regularization of positive currents (Demailly), one proves that existence of **Kähler current** (positive, closed current ξ, such that $\xi - \omega$ is positive for some Hermitian form ω) is equivalent to existence of a Kähler form.

Step 3: Existence of a Kähler current is equivalent to non-existence of a positive current ξ which is a limit of dd^c-closed positive forms and equal to an $(1,1)$-part of an exact current.

Step 4: Non-existence of such ξ is implied by dd^c-lemma.

For HKT-manifolds, dd^c-lemma is the only non-trivial step.

Harvey-Lawson for HKT-structures

THEOREM: Let (M,I,J,K,Φ) be an $SL(2,\mathbb{H})$-manifold, not admitting an HKT-metric. **Then M admits a ∂-exact, positive $(2,0)$-current.**

Proof. Step 1: Apply Hahn-Banach separation theorem to the space A of positive, real (2,0)-forms and W of ∂-closed real (2,0)-forms to **obtain a current $\xi \in \Lambda^{2,0}_{\mathbb{R}}(M,I)^*$ which is positive on A (hence, real and positive) and vanishes on W.**

Step 2: Consider the pairing $\langle \eta, \nu\rangle = \int_M \eta\wedge\nu\wedge\overline{\Phi}$ on $(p,0)$-forms. This pairing is compatible with ∂ and ∂_J and allows one to identify the currents $\Lambda^{p,0}_{\mathbb{R}}(M,I)^*$ with $\Lambda^{n-p,0}_{\mathbb{R}}(M,I)\otimes C\infty(M)^*$, where $C\infty(M)^*$ denotes generalized functions. **This identification is compatible with ∂ and ∂_J; cohomology of currents are the same as cohomology of forms.**

Step 3: Since $\langle\xi, W\rangle = 0$, for each η one has $0 = \langle \xi, \partial\eta\rangle = \langle\partial\xi,\eta\rangle$, giving $\partial\xi = 0$. It remains to show that the cohomology class of ξ in $H^2_\partial(\Lambda^{*,0}(M))$ vanishes

Step 4: The Serre's duality gives a non-degenerate pairing $\langle[\xi],[\nu]\rangle \longrightarrow \mathbb{R}$ on cohomology classes in $H^2_\partial(\Lambda^{*,0}(M))$. Since $\langle[\xi],[\nu]\rangle = 0$ for each ∂-closed nu, the cohomology class of ξ also vanishes. ■

HKT metrics from $\partial\partial_J$-lemma

THEOREM: Let (M, I, J, K, Φ) be a compact $SL(2,\mathbb{H})$-manifold. Assume that $\partial\partial_J$-lemma holds on $\Lambda^{2,0}(M)$. **Then M is HKT.**

Proof: Indeed, if M is not HKT, M admits a ∂-exact positive, real $(2,0)$-current ξ. By $\partial\partial_J$-lemma this current would be $\partial\partial_J$-exact: $\xi = \partial\partial_J f$. Then f is a globally defined $\mathbb{H}$-plurisubharmonic function, hence subharmonic, hence constant. ■

To finish the proof of main theorem, it remains to prove the $\partial\partial_J$-lemma for even $h^1(\mathcal{O}_M)$.

APPENDIX: $\partial\partial_J$-lemma for HKT manifolds

Quaternionic Gauduchon metrics

DEFINITION: A Hermitian metric ω on a complex n manifold is called **Gauduchon** if $\partial\bar\partial\omega^{n-1} = 0$.

THEOREM: (P. Gauduchon, 1978) Let M be a compact, complex manifold, and h a Hermitian form. **Then there exists a Gauduchon metric conformally equivalent** to h, and it is unique, up to a constant multiplier.

DEFINITION: A quaternionic Hermitian form g in a hypercomplex manifold M, $\dim_{\mathbb{H}} M = n.$ is called **quaternionic Gauduchon**, if $\partial\partial_J\Omega^{n-1} = 0$, where $\Omega = \omega_J + \sqrt{-1}\,\omega_K$ is the corresponding positive (2,0)-form.

PROPOSITION: Let (M, I, J, K, Φ) be an $SL(n,\mathbb{H})$-manifold equipped with a quaternionic Hermitian form g, and $|\Phi|^2 := \Phi\wedge\overline{\Phi}/\omega_I^{2n}$. **Then g is quaternionic Gauduchon if and only if the Hermitian metric $|\Phi|^{-1}g$ is Gauduchon on (M, I).**

Proof: A simple linear algebra argument, left as an exercise. ∎

COROLLARY: Quaternionic Gauduchon metrics always exist.

Surjectivity of $f \longrightarrow \Omega^{n-1} \wedge \partial\partial_J f$.

Theorem (*): $(M, I, J, K, \Omega, \Phi)$ be a compact quaternionic Hermitian $SL(n,\mathbb{H})$-manifold. Assume that Ω is $\mathbb{H}$-Gauduchon. Consider the map $D:\ C^\infty M \longrightarrow \Lambda^{4n}(M)$,

$$D(f) = \partial\partial_J f \wedge \Omega^{n-1} \wedge \Phi.$$

Then D induces a bijection between $C^\infty M/const$ and the space of all exact $4n$-forms on M.

Proof. Step 1: Clearly, D is elliptic, and has index 0, because it has the same symbol as Laplacian which is self-adjoint.

Step 2: Hopf maximum principle implies that $\ker D = const$. Therefore, $\operatorname{coker} D$ is 1-dimensional. It remains to show that $\operatorname{im} D$ consists of exact $4n$-forms.

Step 3:

$$\int_M \partial\partial_J f \wedge \Omega^{n-1} \wedge \Phi = -\int_M f \wedge \partial\partial_J(\Omega^{n-1}) \wedge \Phi = 0$$

because Ω is $\mathbb{H}$-Gauduchon. ■

Quaternionic Aeppli and Bott-Chern cohomology.

DEFINITION: Let (M,I,J,K) be a hypercomplex manifold. Define **quaternionic Bott-Chern cohomology** as

$$H^p_{BC}(M) := \frac{\ker\partial \cap \ker\partial_J\Big|_{\Lambda^{p,0}(M)}}{\partial\partial_J(\Lambda^{p-2,0}(M))},$$

and **quaternionic Aeppli cohomology** as

$$H^p_{AE}(M) := \frac{\ker\partial \ker\partial_J\Big|_{\Lambda^{p,0}(M)}}{\partial(\Lambda^{p-2,0}(M)) + \partial_J(\Lambda^{p-2,0}(M))}.$$

REMARK: These spaces are finite-dimensional. Moreover, $H^p_{BC}(M)$ is dual to $H^{2n-p}_{AE}(M)$. The proof is the same as for the usual Bott-Chern and Aeppli cohomology.

CLAIM: **$\partial\partial_J$ lemma for $\Lambda^{2,0}$ is equivalent to vanishing of the map $\partial: H^1_{AE}(M) \longrightarrow H^2_{BC}(M)$.** ■

Degree map for Aeppli cohomology

DEFINITION: Let $(M, I, J, K, \Omega, \Phi)$ be a compact quaternionic Gauduchon $SL(n, \mathbb{H})$-manifold. Consider the map $\deg H^1_{AE}(M) \longrightarrow \mathbb{C}$ putting α to $\int \partial\alpha \wedge \Omega^{n-1} \wedge \overline{\Phi}$. Since Ω is $\mathbb{H}$-Gauduchon, $\deg \alpha$ is independent from the choince of α in its cohomology class.

THEOREM: Let $(M, I, J, K, \Omega, \Phi)$ be a compact quaternionic Gauduchon $SL(2, \mathbb{H})$-manifold. Then the sequence $0 \longrightarrow H^1_\partial(M) \longrightarrow H^1_{AE}(M) \stackrel{\deg}{\longrightarrow} \mathbb{C}$ is exact.

Proof. Step 1: Let $\alpha \in \ker \deg$. By Theorem (*), there exists $f \in C^\infty M$ such that $(\partial\alpha + \partial\partial_J f) \wedge \Omega \wedge \overline{\Phi} = 0$, equivalently $(\partial\alpha + \partial\partial_J f) \wedge \Omega = 0$. Replacing α by $\alpha + \partial_J f$ in the same cohomology class, **we may assume that $\partial\alpha \wedge \Omega = 0$.**

Step 2: Since $\partial\alpha$ is primitive, one has $\int_M \partial\alpha \wedge \partial_J\alpha \wedge \overline{\Phi} = \|\partial\alpha\|^2$ by a quaternionic version of Hodge-Riemann relations.

Step 3: However, $\|\partial\alpha\|^2 = \int_M \partial\alpha \wedge \partial_J\alpha \wedge \overline{\Phi} = -\int_M \partial\partial_J\alpha \wedge \alpha \wedge \overline{\Phi} = 0$, hence $\partial\alpha = 0$. ■

The proof of $\partial\partial_J$-lemma

THEOREM: Let (M,I,J,K,Φ) be a compact $SL(2,\mathbb{H})$-manifold. **Then $\partial\partial_J$-lemma holds on $\Lambda^{2,0}(M)$ if and only if $h^1(\mathcal{O}_M)$ is even.**

Proof. Step 1: Clearly, $\partial\partial_J$-lemma is equivalent to vanishing of $\partial : H^1_{AE}(M) \longrightarrow H^2_{BC}(M)$, but the kernel of this map is $\ker \deg$, hence it suffices to show that $\deg = 0$.

Step 2: Since J defines quaternionic structure on $H^1_{AE}(M)$, this space is even-dimensional. Now, from the exact sequence

$$0 \longrightarrow H^1_{\partial}(M) \longrightarrow H^1_{AE}(M) \xrightarrow{\deg} \mathbb{C},$$

we obtain that $\deg = 0$ whenever $H^1_{\partial}(M)$ is even-dimensional. ∎