

Hypercomplex manifolds of quaternionic dimension 2 and HKT-structures

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Complex and Riemannian Geometry

"Complex Geometry and Lie Groups"

Torino, 16-20 June 2014

Joint work with Geo Grantcharov and Mehdi Lejmi

Hypercomplex manifolds

DEFINITION: Let M be a smooth manifold equipped with endomorphisms $I, J, K : TM \rightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = IJK = -\text{Id}$. Suppose that I, J, K are integrable almost complex structures. Then (M, I, J, K) is called **a hypercomplex manifold**.

THEOREM: (Obata, 1955) On any hypercomplex manifold **there exists a unique torsion-free connection ∇ such that $\nabla I = \nabla J = \nabla K$.**

DEFINITION: Such a connection is called **the Obata connection**.

REMARK: The holonomy of Obata connection lies in $GL(n, \mathbb{H})$.

REMARK: A torsion-free connection ∇ on M with $\text{Hol}(\nabla) \subset GL(n, \mathbb{H})$ defines a hypercomplex structure on M .

Examples of hypercomplex manifolds

EXAMPLE: A **Hopf surface** $M = \mathbb{H} \setminus 0 / \mathbb{Z} \cong S^1 \times S^3$. The holonomy of Obata connection $\mathcal{H}ol(M) = 0$.

EXAMPLE: **Compact holomorphically symplectic manifolds are hyperkähler (by Calabi-Yau theorem)**, hence hypercomplex. Here $\mathcal{H}ol(M) \subset Sp(n)$ **(this is equivalent to being hyperkähler)**.

PROPOSITION: A compact hypercomplex manifold (M, I, J, K) with (M, I) of Kähler type also admits a hyperkähler structure.

REMARK: In dimension 1, compact hypercomplex manifolds are classified (C. P. Boyer, 1988). This is the complete list: **torus, K3 surface, Hopf surface**.

Examples of hypercomplex manifolds (2)

EXAMPLE: The Lie groups

$$\begin{aligned} &SU(2l+1), \quad T^1 \times SU(2l), \quad T^l \times SO(2l+1), \\ &T^{2l} \times SO(4l), \quad T^l \times Sp(l), \quad T^2 \times E_6, \\ &T^7 \times E^7, \quad T^8 \times E^8, \quad T^4 \times F_4, \quad T^2 \times G_2. \end{aligned}$$

Some other homogeneous spaces (D. Joyce and physicists Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen). **Holonomy unknown (but likely $GL(n, \mathbb{H})$).**

THEOREM: (A. Soldatenkov)

Holonomy of Obata connection on $SU(3)$ is $GL(2, \mathbb{H})$.

EXAMPLE: Many **nilmanifolds** (quotients of a nilpotent Lie group by a cocompact lattice) admit hypercomplex structures. **In this case $\text{Hol}(M) \subset SL(n, \mathbb{H})$.**

Quaternionic Hermitian structures

DEFINITION: Let (M, I, J, K) be a hypercomplex manifold, and g a Riemannian metric. We say that g is **quaternionic Hermitian** if I, J, K are orthogonal with respect to g .

CLAIM: Quaternionic Hermitian metrics always exist.

Proof: Take any Riemannian metric g and **consider its average** $\text{Av}_{SU(2)} g$ with respect to $SU(2) \subset \mathbb{H}^*$. ■

Given a quaternionic Hermitian metric g on (M, I, J, K) , consider its Hermitian forms

$$\omega_I(\cdot, \cdot) = g(\cdot, I\cdot), \omega_J, \omega_K$$

(real, but *not closed*). Then $\Omega = \omega_J + \sqrt{-1}\omega_K$ is of Hodge type $(2,0)$ with respect to I .

If $d\Omega = 0$, (M, I, J, K, g) is hyperkähler (this is one of the definitions).

Consider a weaker condition:

$$\partial\Omega = 0, \quad \partial : \Lambda^{2,0}(M, I) \longrightarrow \Lambda^{3,0}(M, I)$$

HKT structures

DEFINITION: (Howe, Papadopoulos, 1998)

Let (M, I, J, K) be a hypercomplex manifold, g a quaternionic Hermitian metric, and $\Omega = \omega_J + \sqrt{-1}\omega_K$ the corresponding $(2, 0)$ -form. We say that g is **HKT (“hyperkähler with torsion”)** if $\partial\Omega = 0$.

HKT-metrics play in hypercomplex geometry the same role as Kähler metrics play in complex geometry.

1. **They admit a smooth potential** (locally). There is a notion of an “HKT-class” (similar to Kähler class) in a certain finite-dimensional cohomology group. Two metrics in the same HKT-class differ by a potential, which is a function.
2. When (M, I) has trivial canonical bundle, **a version of Hodge theory is established** giving an $\mathfrak{sl}(2)$ -action on holomorphic cohomology $H^*(M, \mathcal{O}_{(M, I)})$ and analogue of Hodge decomposition and dd^c -lemma.
3. **Not all compact hypercomplex manifolds are HKT** (Fino, Grantcharov).

Buchsdahl-Lamari theorem and its hypercomplex analogue

THEOREM: (Buchsdahl-Lamari)

Let M be a compact complex surface. Then M is Kähler if and only if $b_1(M)$ is even.

DEFINITION: A compact hypercomplex manifold with Obata holonomy in $SL(n, \mathbb{H})$ is called $SL(n, \mathbb{H})$ -manifold.

THEOREM: (Grantcharov-Lejmi-V.)

Let M be an $SL(2, \mathbb{H})$ -manifold. Then M is HKT if and only if $h^1(\mathcal{O}_{M,I})$ is even.

REMARK: Using the Hodge decomposition on $H^*(\mathcal{O}_{M,I})$, one can show that $h^1(\mathcal{O}_{M,I})$ is even for any $SL(n, \mathbb{H})$ -manifold admitting an HKT structure.

Plan of this talk:

1. Introduce Hodge theory on hypercomplex manifolds.
 - a. HKT-structures.
 - b. Canonical bundle.
 - c. Quaternionic Dolbeault complex.
 - d. Laplacians and cohomology.
2. Explain Harvey-Lawson and Lamari's ideas used in the proof of BuchsdaHL-Lamari theorem.
3. Prove quaternionic analogue of dd^c -lemma for manifolds with even $h^1(\mathcal{O}_{M,I})$.

HKT potential

Defining Kähler metric via Kähler potentials: A Kähler metric on (M, I) is one which is locally given as

$$g(\cdot, \cdot) = \sqrt{-1} \partial \bar{\partial} \varphi(\cdot, I \cdot)$$

where φ is a function called **a Kähler potential**.

Defining HKT metric through HKT potentials: An HKT metric on (M, I) is one which is locally given as

$$g(\cdot, \cdot) = D(\varphi), \quad \text{where } D(\varphi) := \text{Av}_{SU(2)}(\sqrt{-1} \partial \bar{\partial} \varphi(\cdot, I \cdot))$$

and φ is a function called **an HKT potential**.

THEOREM: (Banos-Swann)

This definition is equivalent to the usual one.

DEFINITION: A function which is an HKT potential of some HKT metric is called **strictly \mathbb{H} -plurisubharmonic**, or \mathbb{H} -psh.

REMARK: For any \mathbb{H} -psh function φ , **φ is subharmonic** with respect to any quaternionic Hermitian metric. Therefore, **there are no globally defined \mathbb{H} -psh functions** on compact manifolds.

HKT-forms

DEFINITION: Let g be an HKT metric. The corresponding $(2, 0)$ -form $\Omega = \omega_J + \sqrt{-1} \omega_K$ is called **an HKT-form**.

CLAIM: Consider the multiplicative action of J on $\Lambda^*(M)$. **Then J maps $\Lambda^{p,q}(M)$ to $\Lambda^{q,p}(M)$.**

Proof: I and J anticommute. ■

DEFINITION: A $(2, 0)$ -form Ω on (M, I) is called **real** if $J(\Omega) = \bar{\Omega}$ and **strictly positive** if $\Omega(x, J(\bar{x})) > 0$ for each non-zero $x \in T_I^{1,0}(M)$.

CLAIM: **Any HKT-form is strictly positive and real.** Moreover, **any ∂ -closed strictly positive real form $\Omega \in \Lambda_I^{2,0}(M)$ defines an HKT-metric $g(x, y) := \Omega(x, J(\bar{y}))$.**

Canonical bundle of a hypercomplex manifold.

0. Quaternionic Hermitian structure always exists.
1. **Complex dimension is even.**
2. **The canonical line bundle $\Lambda^{n,0}(M, I)$ of (M, I) is always trivial topologically.** Indeed, a non-degenerate section of canonical line bundle is provided by top power of a form Ω associated with some quaternionic Hermitian structure. In particular, $c_1(M, I) = 0$.
3. Canonical bundle **is non-trivial holomorphically** in many cases. However, **when M is a nilmanifold**, $\Lambda^{n,0}(M, I)$ is trivial, and holonomy of Obata connection lies in $SL(n, \mathbb{H})$ (Barberis-Dotti-V., 2007)
4. If $\mathcal{H}ol(M)$ lies in $SL(n, \mathbb{H})$, canonical bundle is trivial. The converse is true when M is compact and HKT (V., 2004): **an HKT manifold with holomorphically trivial canonical bundle satisfies $\mathcal{H}ol(M) \subset SL(n, \mathbb{H})$.**

$SU(2)$ -action on $\Lambda^*(M)$

The group $SU(2)$ of unitary quaternions acts on TM , because quaternion algebra acts. By multilinearity, this action is extended to $\Lambda^*(M)$.

1. The Hodge decomposition $\Lambda^*(M) = \bigoplus \Lambda^{p,q}(M)$ is recovered from this $SU(2)$ -action. **“Hypercomplex analogue of the Hodge decomposition”**.

2. $\langle \omega_I, \omega_J, \omega_K \rangle$ is an irreducible 3-dimensional representation of $SU(2)$, for any quaternionic Hermitian structure (“representation of weight 2”).

WEIGHT of a representation.

We say that an irreducible $SU(2)$ -representation W **has weight** i if $\dim W = i + 1$. A representation is said to be **pure of weight** i if all its irreducible components have weight i . If all irreducible components of a representation W_1 have weight $\leq i$, we say that W_1 **is a representation of weight** $\leq i$. In a similar fashion one defines representations of weight $\geq i$.

Quaternionic Dolbeault algebra

The weight is multiplicative, in the following sense: a tensor product of representations of weights $\leq i$ and $\leq j$ has weight $\leq i + j$.

Clearly, $\Lambda^1(M)$ has weight 1. Therefore, $\Lambda^i(M)$ **has weight** $\leq i$.

Let $V^i \subset \Lambda^i(M)$ be the maximal $SU(2)$ -invariant subspace of weight $< i$.

By multiplicativity, $V^* = \bigoplus_i V^i$ **is an ideal in** $\Lambda^*(M)$. We also have $V^i = \Lambda^i(M)$ for $i > 2n$. Also, $dV^i \subset V^{i+1}$, hence $V^* \subset \Lambda^*(M)$ is a differential ideal in $(\Lambda^*(M), d)$.

Denote by $(\Lambda_+^*(M), d_+)$ **the quotient algebra** $\Lambda^*(M)/V^*$.

We call it **the quaternionic Dolbeault algebra (qD-algebra)** of M .

A similar construction was given by Salamon in a more general situation.

The Hodge decomposition of quaternionic Dolbeault algebra.

The Hodge decomposition is induced from the $SU(2)$ -action, hence it is **compatible with weights**: $\Lambda_{+}^i(M) = \bigoplus_{p+q=i} \Lambda_{+,I}^{p,q}(M)$.

Let $\sqrt{-1}\mathcal{I}$ be an element of the Lie algebra $\mathfrak{su}(2) \otimes \mathbb{C}$ acting as $\sqrt{-1}(p-q)$ on $\Lambda^{p,q}(M)$. This vector generates the Cartan algebra of $\mathfrak{su}(2)$. The $\mathfrak{su}(2)$ -action induces an isomorphism of $\Lambda_{+,I}^{p,q}(M)$ for all $\{p, q \mid p+q = k, p, q \geq 0\}$. This gives

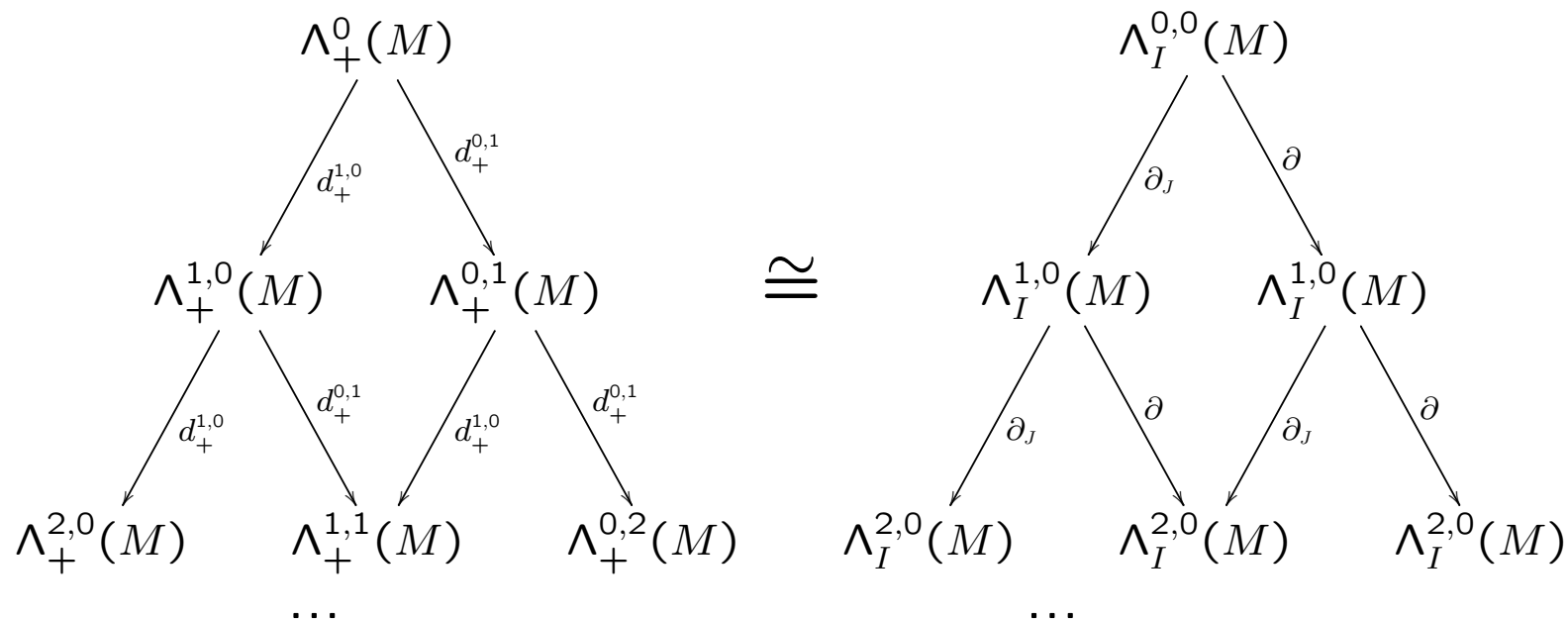
Theorem: $\Lambda_{+,I}^{p,q}(M) \cong \Lambda^{0,p+q}(M, I)$.

This isomorphism is provided by the $\mathfrak{su}(2) \otimes \mathbb{C}$ -action.

Differentials in the qD -complex

We extend $J : \Lambda^1(M) \rightarrow \Lambda^1(M)$ to $\Lambda^*(M)$ by multiplicativity. Since I and J anticommute on $\Lambda^1(M)$, **we have** $J(\Lambda^{p,q}(M, I)) = \Lambda^{q,p}(M, I)$.

Denote by $\partial_J : \Lambda^{p,0}(M, I) \rightarrow \Lambda^{p,0}(M, I)$ **the operator** $J \circ \bar{\partial} \circ J$, where $\bar{\partial} : \Lambda^{0,p}(M, I) \rightarrow \Lambda^{0,p}(M, I)$ is the standard Dolbeault differential. Then ∂, ∂_J anticommute. Moreover, **there exists a multiplicative isomorphism of bicomplexes.**



Potentials for HKT-metrics

A quaternionic Hermitian metric **can be recovered from the corresponding $(2, 0)$ -form**: $\omega_I(x, \bar{y}) = \frac{1}{2}\Omega(x, J(\bar{y}))$, where $x, y \in T^{1,0}(M)$. The HKT-structures uniquely correspond to $(2, 0)$ -forms which are

1. Real: $J(\Omega) = \bar{\Omega}$

2. Closed: $\partial\Omega = 0$.

2. Positive: $\Omega(x, J(\bar{x})) > 0$, for any non-zero $x \in T^{1,0}(M)$

Locally, any HKT-metric is given by a potential: $\Omega = \partial\bar{\partial}_J\varphi$ where φ is a smooth function.

Any convex, and any strictly plurisubharmonic function is a potential of some HKT-structure. Therefore, HKT-structures locally always exist (Grantcharov, Poon).

DEFINITION: Quaternionic Hessian of $f \in C^\infty M$ is a form $x, y \longrightarrow \partial\bar{\partial}_J\varphi(x, J(\bar{x}))$. It is equal to the usual Hessian averaged with $SU(2)$. A function is **quaternionic plurisubharmonic** if its \mathbb{H} -Hessian is positive; equivalently, if $\partial\bar{\partial}_J f$ is a positive $(2, 0)$ -form.

Hodge theory on HKT-manifolds with holonomy in $SL(n, \mathbb{H})$

DEFINITION: Let Φ be a non-degenerate, real, Obata-parallel section of $\Lambda^{n,0}(M, I)$. Then (M, J, J, K, Φ) is called **an $SL(n, \mathbb{H})$ -manifold**.

DEFINITION: Let M be a compact HKT-manifold with holonomy in $SL(n, \mathbb{H})$, and $\Delta_{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. Then $\ker \Delta_{\bar{\partial}}|_{\Lambda^{0,*}(M)} = H^*(M, \mathcal{O}_{(M,I)})$.

THEOREM: $\Delta_{\bar{\partial}}$ commutes with the multiplication by the HKT-form $\bar{\Omega}$, and with the operator $\eta \rightarrow J(\bar{\eta})$. In particular, **there is a Lefschetz-like $\mathfrak{sl}(2)$ -action on $H^*(M, \mathcal{O}_{(M,I)})$** .

REMARK: To simplify notation, it is more convenient to consider the ∂ and ∂_J -Laplacian, and to identify $H^*(M, \mathcal{O}_{(M,I)})$ with the cohomology of ∂ .

THEOREM: The Laplacians Δ_{∂} and Δ_{∂_J} are equal. In particular, $\eta \rightarrow J(\bar{\eta})$ defines a complex structure on Δ_{∂} -harmonic $(k, 0)$ -forms for odd k , and real structure for even k .

Theorem (“ $\partial\partial_J$ -lemma”) Let η be a ∂_J -closed, ∂ -exact form on an HKT $SL(n, \mathbb{H})$ -manifold. **Then η is $\partial\partial_J$ -exact: $\eta \in \text{im } \partial\partial_J\eta$.**

Hahn-Banach separation theorem and its applications

THEOREM: Let V be a locally convex topological vector space, $W \subset V$ a closed subspace, and $A \subset V$ an open, convex subset, not intersecting W . **Then there exists a continuous linear functional $\xi \in V^*$ vanishing on W and positive on A .**

THEOREM: (Harvey, Lawson):

Let M be a compact complex non-Kähler manifold. Then there exists a positive $(n-1, n-1)$ -current ξ which is a $(n-1, n-1)$ -part of an exact current.

Idea of a proof: Hahn-Banach separation theorem is applied to the set A of strictly positive $(1, 1)$ -forms, and the set W of closed $(1, 1)$ -forms, obtaining a current $\xi \in D^{n-1, n-1}(M) = \Lambda^{1,1}(M)^*$ positive on A (that is, positive) and vanishing on W . The later condition (after some simple cohomological manipulations) becomes “ $(n-1, n-1)$ -part of an exact current”. ■

Lamari's proof of Buchsdaahl-Lamari theorem

THEOREM: (Buchsdaahl-Lamari)

Let M be a compact complex surface. Then M is Kähler if and only if $b_1(M)$ is even.

Scheme of Lamari's proof:

Step 1: Evenness of $b_1(M)$ is equivalent to dd^c -lemma.

Step 2: Using regularization of positive currents (Demailly), one proves that existence of **Kähler current** (positive, closed current ξ , such that $\xi - \omega$ is positive for some Hermitian form ω) is equivalent to existence of a Kähler form.

Step 3: Existence of a Kähler current is equivalent to non-existence of a positive current ξ which is a limit of dd^c -closed positive forms and equal to an $(1, 1)$ -part of an exact current.

Step 4: Non-existence of such ξ is implied by dd^c -lemma.

For HKT-manifolds, dd^c -lemma is the only non-trivial step.

Harvey-Lawson for HKT-structures

THEOREM: Let (M, I, J, K, Φ) be an $SL(2, \mathbb{H})$ -manifold, not admitting an HKT-metric. **Then M admits a ∂ -exact, positive $(2, 0)$ -current.**

Proof. Step 1: Apply Hahn-Banach separation theorem to the space A of positive, real $(2, 0)$ -forms and W of ∂ -closed real $(2, 0)$ -forms to **obtain a current $\xi \in \Lambda_{\mathbb{R}}^{2,0}(M, I)^*$ which is positive on A (hence, real and positive) and vanishes on W .**

Step 2: Consider the pairing $\langle \eta, \nu \rangle = \int_M \eta \wedge \nu \wedge \bar{\Phi}$ on $(p, 0)$ -forms. This pairing is compatible with ∂ and ∂_J and allows one to identify the currents $\Lambda_{\mathbb{R}}^{p,0}(M, I)^*$ with $\Lambda_{\mathbb{R}}^{n-p,0}(M, I) \otimes C_{\infty}(M)^*$, where $C_{\infty}(M)^*$ denotes generalized functions. **This identification is compatible with ∂ and ∂_J ; cohomology of currents are the same as cohomology of forms.**

Step 3: Since $\langle \xi, W \rangle = 0$, for each η one has $0 = \langle \xi, \partial \eta \rangle = \langle \partial \xi, \eta \rangle$, giving $\partial \xi = 0$. It remains to show that the cohomology class of ξ in $H_{\partial}^2(\Lambda^{*,0}(M))$ vanishes

Step 4: The Serre's duality gives a non-degenerate pairing $\langle [\xi], [\nu] \rangle \rightarrow \mathbb{R}$ on cohomology classes in $H_{\partial}^2(\Lambda^{*,0}(M))$. Since $\langle [\xi], [\nu] \rangle = 0$ for each ∂ -closed ν , the cohomology class of ξ also vanishes. ■

HKT metrics from $\partial\bar{\partial}_J$ -lemma

THEOREM: Let (M, I, J, K, Φ) be a compact $SL(2, \mathbb{H})$ -manifold. Assume that $\partial\bar{\partial}_J$ -lemma holds on $\Lambda^{2,0}(M)$. **Then M is HKT.**

Proof: Indeed, if M is not HKT, M admits a ∂ -exact positive, real $(2, 0)$ -current ξ . By $\partial\bar{\partial}_J$ -lemma this current would be $\partial\bar{\partial}_J$ -exact: $\xi = \partial\bar{\partial}_J f$. Then f is a globally defined \mathbb{H} -plurisubharmonic function, hence subharmonic, hence constant. ■

To finish the proof of main theorem, it remains to prove the $\partial\bar{\partial}_J$ -lemma for even $h^1(\mathcal{O}_M)$.

Quaternionic Gauduchon metrics

DEFINITION: A Hermitian metric ω on a complex n manifold is called **Gauduchon** if $\partial\bar{\partial}\omega^{n-1} = 0$.

THEOREM: (P. Gauduchon, 1978) Let M be a compact, complex manifold, and h a Hermitian form. **Then there exists a Gauduchon metric conformally equivalent** to h , and it is unique, up to a constant multiplier.

DEFINITION: A quaternionic Hermitian form g in a hypercomplex manifold M , $\dim_{\mathbb{H}} M = n$. is called **quaternionic Gauduchon**, if $\partial\partial_J\Omega^{n-1} = 0$, where $\Omega = \omega_J + \sqrt{-1}\omega_K$ is the corresponding positive $(2,0)$ -form.

PROPOSITION: Let (M, I, J, K, Φ) be an $SL(n, \mathbb{H})$ -manifold equipped with a quaternionic Hermitian form g , and $|\Phi|^2 := \Phi \wedge \bar{\Phi} / \omega_I^{2n}$. **Then g is quaternionic Gauduchon if and only if the Hermitian metric $|\Phi|^{-1}g$ is Gauduchon on (M, I) .**

Proof: A simple linear algebra argument, left as an exercise. ■

COROLLARY: **Quaternionic Gauduchon metrics always exist.**

Surjectivity of $f \longrightarrow \Omega^{n-1} \wedge \partial\bar{\partial}_J f$.

Theorem (*): $(M, I, J, K, \Omega, \Phi)$ be a compact quaternionic Hermitian $SL(n, \mathbb{H})$ -manifold. Assume that Ω is \mathbb{H} -Gauduchon. Consider the map $D : C^\infty M \longrightarrow \Lambda^{4n}(M)$,

$$D(f) = \partial\bar{\partial}_J f \wedge \Omega^{n-1} \wedge \Phi.$$

Then D induces a bijection between $C^\infty M / \text{const}$ and the space of all exact $4n$ -forms on M .

Proof. Step 1: Clearly, D is elliptic, and has index 0, because it has the same symbol as Laplacian which is self-adjoint.

Step 2: Hopf maximum principle implies that $\ker D = \text{const}$. Therefore, $\text{coker } D$ is 1-dimensional. It remains to show that $\text{im } D$ consists of exact $4n$ -forms.

Step 3:

$$\int_M \partial\bar{\partial}_J f \wedge \Omega^{n-1} \wedge \Phi = - \int_M f \wedge \partial\bar{\partial}_J(\Omega^{n-1}) \wedge \Phi = 0$$

because Ω is \mathbb{H} -Gauduchon. ■

Quaternionic Aeppli and Bott-Chern cohomology.

DEFINITION: Let (M, I, J, K) be a hypercomplex manifold. Define **quaternionic Bott-Chern cohomology** as

$$H_{BC}^p(M) := \frac{\ker \partial \cap \ker \partial_J|_{\Lambda^{p,0}(M)}}{\partial \partial_J(\Lambda^{p-2,0}(M))},$$

and **quaternionic Aeppli cohomology** as

$$H_{AE}^p(M) := \frac{\ker \partial \ker \partial_J|_{\Lambda^{p,0}(M)}}{\partial(\Lambda^{p-2,0}(M)) + \partial_J(\Lambda^{p-2,0}(M))}.$$

REMARK: These spaces are finite-dimensional. Moreover, $H_{BC}^p(M)$ is dual to $H_{AE}^{2n-p}(M)$. The proof is the same as for the usual Bott-Chern and Aeppli cohomology.

CLAIM: $\partial \partial_J$ lemma for $\Lambda^{2,0}$ is equivalent to vanishing of the map $\partial : H_{AE}^1(M) \rightarrow H_{BC}^2(M)$. ■

Degree map for Aeppli cohomology

DEFINITION: Let $(M, I, J, K, \Omega, \Phi)$ be a compact quaternionic Gauduchon $SL(n, \mathbb{H})$ -manifold. Consider the map $\text{deg} : H_{AE}^1(M) \rightarrow \mathbb{C}$ putting α to $\int \partial\alpha \wedge \Omega^{n-1} \wedge \bar{\Phi}$. Since Ω is \mathbb{H} -Gauduchon, $\text{deg} \alpha$ is independent from the choice of α in its cohomology class.

THEOREM: Let $(M, I, J, K, \Omega, \Phi)$ be a compact quaternionic Gauduchon $SL(2, \mathbb{H})$ -manifold. Then the sequence $0 \rightarrow H_{\partial}^1(M) \rightarrow H_{AE}^1(M) \xrightarrow{\text{deg}} \mathbb{C}$ is exact.

Proof. Step 1: Let $\alpha \in \ker \text{deg}$. By Theorem (*), there exists $f \in C^\infty M$ such that $(\partial\alpha + \partial\partial_J f) \wedge \Omega \wedge \bar{\Phi} = 0$, equivalently $(\partial\alpha + \partial\partial_J f) \wedge \Omega = 0$. Replacing α by $\alpha + \partial_J f$ in the same cohomology class, **we may assume that $\partial\alpha \wedge \Omega = 0$.**

Step 2: Since $\partial\alpha$ is primitive, one has $\int_M \partial\alpha \wedge \partial_J \alpha \wedge \bar{\Phi} = \|\partial\alpha\|^2$ by a quaternionic version of Hodge-Riemann relations.

Step 3: However, $\|\partial\alpha\|^2 = \int_M \partial\alpha \wedge \partial_J \alpha \wedge \bar{\Phi} = -\int_M \partial\partial_J \alpha \wedge \alpha \wedge \bar{\Phi} = 0$, hence $\partial\alpha = 0$. ■

The proof of $\partial\bar{\partial}_J$ -lemma

THEOREM: Let (M, I, J, K, Φ) be a compact $SL(2, \mathbb{H})$ -manifold. **Then $\partial\bar{\partial}_J$ -lemma holds on $\Lambda^{2,0}(M)$ if and only if $h^1(\mathcal{O}_M)$ is even.**

Proof. Step 1: Clearly, $\partial\bar{\partial}_J$ -lemma is equivalent to vanishing of $\partial : H_{AE}^1(M) \longrightarrow H_{BC}^2(M)$, but the kernel of this map is $\ker \text{deg}$, hence it suffices to show that $\text{deg} = 0$.

Step 2: Since J defines quaternionic structure on $H_{AE}^1(M)$, this space is even-dimensional. Now, from the exact sequence

$$0 \longrightarrow H_{\bar{\partial}}^1(M) \longrightarrow H_{AE}^1(M) \xrightarrow{\text{deg}} \mathbb{C},$$

we obtain that $\text{deg} = 0$ whenever $H_{\bar{\partial}}^1(M)$ is even-dimensional. ■