Hyperkähler manifolds

lecture 1: Kähler manifolds and holonomy groups

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Complex action on vector spaces

Let V be a vector space over $\mathbb{R}$, and $I: V \longrightarrow V$ an automorphism which satisfies $I^2 = -\operatorname{Id}_V$. **We extend the action of I on the tensor spaces $V \otimes V \otimes ... \otimes V \otimes V^* \otimes V^* \otimes ... \otimes V^*$ by multiplicativity:** $I(v_1 \otimes ... \otimes w_1 \otimes ... \otimes w_n) = I(v_1) \otimes ... \otimes I(w_1) \otimes ... \otimes I(w_n)$.

Trivial observations:

1. **The eigenvalues of I are $\pm\sqrt{-1}$.**

2. **V admits an I-invariant metric g.** Take any metric g_0, and let $g := g_0 + I(g_0)$.

3. **I diagonalizable over $\mathbb{C}$.** Indeed, any orthogonal matrix is diagonalizable.

4. **All eigenvalues of I are equal to $\pm\sqrt{-1}$.** Indeed, $I^2 = -1$.

5. **There are as many $\sqrt{-1}$-eigenvalues as there are $-\sqrt{-1}$-eigenvalues.** Indeed, I is real.

The Hodge decomposition in linear algebra

DEFINITION: The Hodge decomposition $V \otimes_{\mathbb R} \mathbb C := V^{1,0} \oplus V^{0,1}$ is defined in such a way that $V^{1,0}$ is a $\sqrt{-1}$-eigenspace of I, and $V^{0,1}$ a $-\sqrt{-1}$-eigenspace.

REMARK: Let $V_{\mathbb C} := V \otimes_{\mathbb R} \mathbb C$. The Grassmann algebra of skew-symmetric forms $\Lambda^n V_{\mathbb C} := \Lambda^n_{\mathbb R} V \otimes_{\mathbb R} C$ admits a decomposition

$$\Lambda^n V_{\mathbb C} = \bigoplus_{p+q=n} \Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$$

We denote $\Lambda^p V^{1,0} \otimes \Lambda^q V^{0,1}$ by $\Lambda^{p,q} V$. The resulting decomposition $\Lambda^n V_{\mathbb C} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is called **the Hodge decomposition of the Grassmann algebra**.

REMARK: The operator I induces $U(1)$-action on V by the formula $\rho(t)(v) = \cos t \cdot v + \sin t \cdot I(v)$. We extend this action on the tensor spaces by muptiplicativity.

$U(1)$-representations and the weight decomposition

REMARK: Any complex representation W of $U(1)$ is written as a sum of 1-dimensional representations $W_i(p)$, with $U(1)$ acting on each $W_i(p)$ as $\rho(t)(v) = e^{\sqrt{-1}\,pt}(v)$. The 1-dimensional representations are called **weight p representations of $U(1)$.**

DEFINITION: A **weight decomposition** of a $U(1)$-representation W is a decomposition $W = \oplus W^p$, where each $W^p = \oplus_i W_i(p)$ is a sum of 1-dimensional representations of weight p.

REMARK: The Hodge decomposition $\Lambda^n V_{\mathbb{C}} = \bigoplus_{p+q=n} \Lambda^{p,q} V$ is a weight decomposition, with $\Lambda^{p,q}V$ being a weight $p-q$-component of $\Lambda^n V_{\mathbb{C}}$.

REMARK: $V^{p,p}$ is the space of $U(1)$-invariant vectors in $\Lambda^{2p}V$.

Further on, **TM is the tangent bundle on a manifold, and $\Lambda^i M$ the space of differential i-forms.** It is a Grassman algebra on TM

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

REMARK: The commutator defines a $\mathbb{C}^\infty M$-linear map $N := \Lambda^2(T^{1,0}) \longrightarrow T^{0,1}M$, called **the Nijenhuis tensor** of I. **One can represent N as a section of $\Lambda^{2,0}(M) \otimes T^{0,1}M$.**

Exercise: Prove that $\mathbb{C}P^n$ **is a complex manifold,** in the sense of the above definition.

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

REMARK: It is $U(1)$-invariant, hence **of Hodge type (1,1)**.

DEFINITION: A complex Hermitian manifold (M, I, ω) is called **Kähler** if $d\omega = 0$. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M, and ω **the Kähler form**.

Examples of Kähler manifolds.

Definition: Let $M=\mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$.** The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$.

Remark: For any $x\in\mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_x\mathbb{C}P^n=\mathbb{C}^n$ is $U(n)$-invariant, hence unique up to a constant.

Claim: Fubini-Study form is Kähler. Indeed, $d\omega|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish, because $-\operatorname{Id}\in U(n)$

REMARK: The same argument works for all symmetric spaces.

Corollary: Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler. Indeed, a restriction of a closed form is again closed.

Connections

Notation: Let M be a smooth manifold, TM its tangent bundle, $\Lambda^i M$ the bundle of differential i-forms, $C^\infty M$ the smooth functions. **The space of sections of a bundle B is denoted by B.**

DEFINITION: A **connection** on a vector bundle B is a map $B \xrightarrow{\nabla} \Lambda^1 M \otimes B$ which satisfies

$$\nabla(fb) = df \otimes b + f\nabla b$$

for all $b \in B$, $f \in C^\infty M$.

REMARK: A connection ∇ on B gives a connection $B^* \xrightarrow{\nabla^*} \Lambda^1 M \otimes B^*$ on the dual bundle, by the formula

$$d(\langle b, \beta \rangle) = \langle \nabla b, \beta \rangle + \langle b, \nabla^* \beta \rangle$$

These connections are usually denoted **by the same letter ∇.**

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

Torsion

DEFINITION: A torsion of a connection $\Lambda^1 \xrightarrow{\nabla} \Lambda^1 M \otimes \Lambda^1 M$ is a map $\mathsf{Alt} \circ \nabla - d$, where $\mathsf{Alt}: \Lambda^1 M \otimes \Lambda^1 M \longrightarrow \Lambda^2 M$ is exterior multiplication. It is a map $T_\nabla: \Lambda^1 M \longrightarrow \Lambda^2 M$.

An exercise: Prove that torsion is a $C^\infty M$-linear.

DEFINITION: Let (M, g) be a Riemannian manifold. A connection ∇ is called **orthogonal** if $\nabla(g) = 0$. It is called **Levi-Civita** if it is torsion-free.

THEOREM: ("the main theorem of differential geometry")
For any Riemannian manifold, the Levi-Civita connection exists, and it is unique.

Levi-Civita connection and Kähler geometry

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) **The complex structure I is integrable, and the Hermitian form ω is closed.**

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection.

REMARK: The implication (ii) $\Rightarrow$ (i) is clear. Indeed, $[X, Y] = \nabla_X Y - \nabla_Y X$, hence it is a (1, 0)-vector field when X, Y are of type (1,0), and then I **is integrable**. Also, $d\omega = 0$, **because ∇ is torsion-free,** and $d\omega = \mathrm{Alt}(\nabla\omega)$.

The implication (i) $\Rightarrow$ (ii) is proven by the same argument as used to construct the Levi-Civita connection.

Holonomy group

DEFINITION: (Cartan, 1923) Let (B, ∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla}: B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B, ∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: A bundle is **flat** (has vanishing curvature) **if and only if its restricted holonomy vanishes.**

REMARK: If $\nabla(\varphi) = 0$ for some tensor $\varphi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$, **the holonomy group preserves φ.**

DEFINITION: **Holonomy of a Riemannian manifold** is holonomy of its Levi-Civita connection.

EXAMPLE: Holonomy of a Riemannian manifold lies in $O(T_xM, g|_x) = O(n)$.

EXAMPLE: Holonomy of a Kähler manifold lies in $U(T_xM, g|_x, I|_x) = U(n)$.

REMARK: The holonomy group **does not depend on the choice of a point $x \in M$.**

Curvature of a connection

Let M be a manifold, B a bundle, $\Lambda^i M$ the differential forms, and $\nabla : B \longrightarrow B \otimes \Lambda^1 M$ a connection. We extend ∇ to $B \otimes \Lambda^i M \xrightarrow{\nabla} B \otimes \Lambda^{i+1} M$ in a natural way, using the formula

$$\nabla(b \otimes \eta) = \nabla(b) \wedge \eta + b \otimes d\eta,$$

and define **the curvature Θ_∇** of ∇ as $\nabla \circ \nabla : B \longrightarrow B \otimes \Lambda^2 M$.

CLAIM: This operator is $C^\infty M$-linear.

REMARK: We shall consider Θ_∇ as an element of $\Lambda^2 M \otimes \operatorname{End} B$, that is, an $\operatorname{End} B$-valued 2-form.

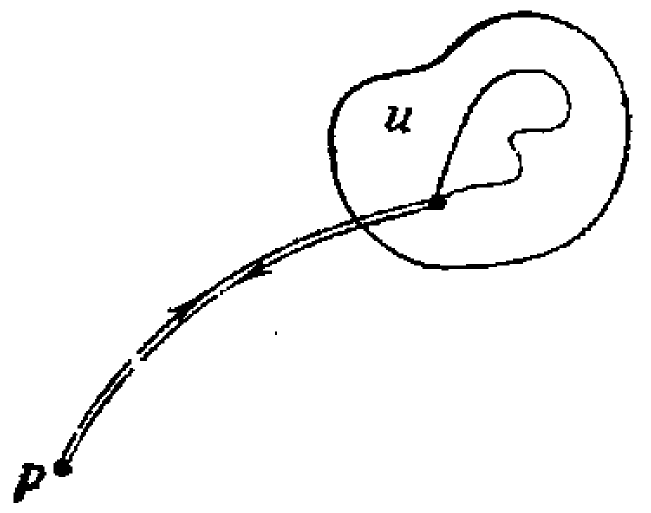
REMARK: Given vector fields $X, Y \in TM$, the curvature can be written in terms of a connection as follows

$$\Theta_\nabla(b) = \nabla_X \nabla_Y b - \nabla_Y \nabla_X B - \nabla_{[X,Y]} b.$$

CLAIM: Suppose that the structure group of B is reduced to its subgroup G, and let ∇ be a connection which preserves this reduction. This is the same as to say that the connection form takes values in $\Lambda^1 \otimes \mathfrak{g}(B)$. **Then Θ_∇ lies in $\Lambda^2 M \otimes \mathfrak{g}(B)$.**

The Lasso lemma

DEFINITION: A **lasso** is a loop of the following form:



The round part is called **a working part** of a loop.

REMARK: ("The Lasso Lemma") Let $\{U_i\}$ be a covering of a manifold, and γ a loop. Then **any contractible loop γ is a product of several lasso, with working part of each inside some U_i.**

The Ambrose-Singer theorem

DEFINITION: Let (B,∇) be a bundle with connection, $\Theta \in \Lambda^2(M)\otimes \operatorname{End}(B)$ its curvature, and $a,b \in T_xM$ tangent vectors. An endomorphism $\Theta(a,b) \in \operatorname{End}(B)|_x$ is called **a curvature element**.

THEOREM: (Ambrose-Singer) The restricted holonomy group of B,∇ at $z \in M$ is a Lie group, **with its Lie algebra generated by all curvature elements $\Theta(a,b) \in \operatorname{End}(B)|_x$ transported to z along all paths.**

REMARK: Its proof follows from the Lasso lemma.

Holonomy representation

DEFINITION: Let (M, g) be a Riemannian manifold, G its holonomy group. A **holonomy representation** is the natural action of G on TM.

THEOREM: (de Rham) Suppose that the holonomy representation is not irreducible: $T_xM = V_1 \oplus V_2$. Then M locally splits as $M = M_1 \times M_2$, with $V_1 = TM_1$, $V_2 = TM_2$.

Proof. Step 1: Using the parallel transform, we extend $V_1 \oplus V_2$ to a **splitting of vector bundles $TM = B_1 \oplus B_2$, preserved by holonomy.**

Step 2: The sub-bundles B_1, $B_2 \subset TM$ **are integrable:** $[B_1, B_1] \subset B_i$ (the Levi-Civita connection is torsion-free)

Step 3: Taking the leaves of these integrable distributions, **we obtain a local decomposition $M = M_1 \times M_2$, with $V_1 = TM_1$, $V_2 = TM_2$.**

Step 4: Since the splitting $TM = B_1 \oplus B_2$ is preserved by the connection, **the leaves M_1, M_2 are totally geodesic.**

Step 5: Therefore, **locally M splits (as a Riemannian manifold)**: $M = M_1 \times M_2$, where M_1, M_2 are any leaves of these foliations. ■

The de Rham splitting theorem

COROLLARY: Let M be a Riemannian manifold, and $\mathcal{H}\mathrm{ol}_0(M) \xrightarrow{\rho} \mathrm{End}(T_xM)$ a reduced holonomy representation. Suppose that ρ is reducible: $T_xM = V_1 \oplus V_2 \oplus ... \oplus V_k$. **Then $G = \mathcal{H}\mathrm{ol}_0(M)$ also splits: $G = G_1 \times G_2 \times ... \times G_k$,** with each G_i acting trivially on all V_j with $j \neq i$.

Proof: Locally, this statement follows from the local splitting of M proven above. To obtain it globally in M, use the Lasso Lemma. ■

THEOREM: (de Rham) A complete, simply connected Riemannian manifold with non-irreducible holonomy **splits as a Riemannian product.**

REMARK: It is easy to find non-complete or non-simply connected counterexamples to de Rham theorem.

THEOREM: (Simons, 1962) Let M be a manifold with irreducible holonomy. **Then either M is locally symmetric, or $\mathcal{H}\mathrm{ol}(M)$ acts transitively on the unit sphere in T_xM.**

Berger's theorem

THEOREM: (Berger's theorem, 1955) Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

REMARK: There is one more group acting transitively on a sphere: $Spin(9)$ acting on $S^{15} \subset \mathbb{R}^{16}$. In 1968, D. Alekseevsky has shown that **a manifold with holonomy $Spin(9)$ is automatically locally symmetric.**

REMARK: A similar list exists for non-orthogonal irreducible holonomy without torsion (Merkulov, Schwachhöfer, 1999).

Chern connection

DEFINITION: Let B be a holomorphic vector bundle, and $\overline{\partial}: B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ an operator mapping $b \otimes f$ to $b \otimes \overline{\partial} f$, where $b \in B$ is a holomorphic section, and f a smooth function. This operator is called **a holomorphic structure operator** on B. **It is correctly defined, because $\overline{\partial}$ is $\mathcal{O}_M$-linear.**

REMARK: A section $b \in B$ is holomorphic iff $\overline{\partial}(b) = 0$

DEFINITION: let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}: B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider the Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$. We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: An Hermitian holomorphic vector bundle is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure.

DEFINITION: A Chern connection on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

Calabi-Yau manifolds

DEFINITION:
A Calabi-Yau manifold is a compact Kaehler manifold with $c_1(M, \mathbb{Z}) = 0$.

DEFINITION: Let (M, I, ω) be a Kaehler n-manifold, and $K(M) := \Lambda^{n,0}(M)$ its **canonical bundle.** We consider $K(M)$ as a colomorphic line bundle, $K(M) = \Omega^n M$. The natural Hermitian metric on $K(M)$ is written as

$$(\alpha, \alpha') \longrightarrow \frac{\alpha \wedge \overline{\alpha}'}{\omega^n}.$$

Denote by Θ_K the curvature of the Chern connection on $K(M)$. The **Ricci curvature** Ric of M is symmetric 2-form $\operatorname{Ric}(x, y) = \Theta_K(x, Iy)$.

DEFINITION: A Kähler manifold is called **Ricci-flat** if its Ricci curvature vanishes.

THEOREM: (Calabi-Yau)
Let (M, I, g) be Calabi-Yau manifold. **Then there exists a unique Ricci-flat Kaehler metric in any given Kaehler class.**

REMARK: Converse is also true: **any Ricci-flat Kähler manifold has a finite covering which is Calabi-Yau.** This is due to Bogomolov.

Bochner's vanishing

THEOREM: (Bochner vanishing theorem) On a compact Ricci-flat Calabi-Yau manifold, **any holomorphic p-form η is parallel** with respect to the Levi-Civita connection: $\nabla(\eta) = 0$.

DEFINITION: A **holomorphic symplectic manifold** is a manifold admitting a non-degenerate, holomorphic symplectic form.

REMARK: A holomorphic symplectic manifold is Calabi-Yau. The top exterior power of a holomorphic symplectic form **is a non-degenerate section of canonical bundle.**

REMARK: Due to Bochner's vanishing, **holonomy of Ricci-flat Calabi-Yau manifold lies in $SU(n)$**, and **holonomy of Ricci-flat holomorphically symplectic manifold lies in $Sp(n)$.** (a group of complex unitary matrices preserving a complex-linear symplectic form).

DEFINITION: A holomorphically symplectic Ricci-flat Kaehler manifold is called **hyperkähler**.

REMARK: Since $Sp(n) = SU(\mathbb{H}, n)$, a **hyperkähler manifold admits quaternionic action in its tangent bundle.**