Hyperkähler SYZ conjecture

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Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I : TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{0,1}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2 y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure I is integrable, and the Hermitian form ω is closed.

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M. The set of all Kähler classes is called **the Kähler cone**.

Hyperkähler manifolds

DEFINITION: A **hypercomplex manifold** is a manifold M equipped with three complex structure operators I, J, K, satisfying **quaternionic relations**

$$IJ = -JI = K, \quad I^2 = J^2 = K^2 = -\operatorname{Id}_{TM}$$

(the last equation is a part of the definition of almost complex structures).

DEFINITION: A **hyperkähler manifold** is a hypercomplex manifold equipped with a metric g which is Kähler with respect to I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $H^1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Remark: A simple hyperkähler manifold is always simply connected (Cheeger-Gromoll theorem).

Further on, all hyperkähler manifolds are assumed to be simple.

Holomorphic Lagrangian fibrations

THEOREM: (Matsushita, 1997)
Let $\pi : M \longrightarrow X$ be a surjective holomorphic map from a hyperkähler manifold M to X, whith $0 < \dim X < \dim M$. **Then $\dim X = 1/2 \dim M$, and the fibers of π are holomorphic Lagrangian** (this means that the symplectic form vanishes on $\pi^{-1}(x)$).

DEFINITION: Such a map is called **holomorphic Lagrangian fibration**.

REMARK: The base of π is conjectured to be rational. Hwang (2007) proved that $X \cong \mathbb{C}P^n$, if it is smooth. Matsushita (2000) proved that it has the same rational cohomology as $\mathbb{C}P^n$.

REMARK: The base of π has a natural flat connection on the smooth locus of π. The combinatorics of this connection can be used to determine the topology of M (Strominger-Yau-Zaslow, Kontsevich-Soibelman).

If we want to learn something about M, it's recommended to start from a holomorphic Lagrangian fibration (if it exists).

The SYZ conjecture

DEFINITION: Let (M, ω) be a Calabi-Yau manifold, Ω the holomorphic volume form, and $Z \subset M$ a real analytic subvariety, Lagrangian with respect to ω. If $\Omega\Big|_Z$ is proportional to the Riemannian volume form, Z is called **special Lagrangian** (SpLag).

(Harvey-Lawson): **SpLag subvarieties minimize Riemannian volume in their cohomology class.** This implies that their moduli are finite-dimensional.

A trivial remark: A holomorphic Lagrangian subvariety of a hyperkähler manifold (M, I) is special Lagrangian on (M, J), where (I, J, K) is a quaternionic structure associated with the hyperkähler structure.

Another trivial remark: A smooth fiber of a Lagrangian fibration has trivial tangent bundle. In particular, **a smooth fiber of a holomorphic Lagrangian fibration is a torus.**

Strominger-Yau-Zaslow, "Mirror symmetry as T-duality" (1997). Any Calabi-Yau manifold admits a Lagrangian fibration with special Lagrangian fibers. Taking its dual fibration, one obtains "the mirror dual" Calabi-Yau manifold.

Ample bundles

REMARK: Let L be a holomorphic line bundle. For any metric on L one associates its Chern connection; the curvature Θ of this connection is a closed, imaginary $(1,1)$-form. If the form $-\sqrt{-1}\,\Theta$ is Kähler, L is called **positive**.

REMARK: This is the usual source of Kähler metrics in complex geometry.

REMARK: **The form $-\sqrt{-1}\,\Theta$ is Kähler if and only if $\Theta(x,\overline{x})>0$ for any non-zero $x\in T^{0,1}(M)$.**

DEFINITION: A holomorphic line bundle is called **ample**, if for a sufficiently big N, the tensor power L^N is generated by global holomorphic sections, without common zeros, and, moreover, the natural map $M\longrightarrow \mathbb{P}H^0(L^N)^*$ is an embedding.

THEOREM: (Kodaira) **Let L be line bundle on a compact complex manifold, with $c_1(L)$ Kähler. Then L is ample.**

Nef classes and semiample bundles

DEFINITION: A cohomology class θ is called **nef** (numerically effective) if it belongs to the closure of the Kähler cone. A holomorphic line bundle L is **nef** if $c_1(L)$ is nef.

DEFINITION: A line bundle is called **semiample** if L^N is generated by its holomorphic sections, which have no common zeros.

REMARK: From semiampleness it obviously follows that L is nef. Indeed, let $\pi: M \longrightarrow \mathbb{P}H^0(L^N)^*$ the the standard map. Since sections of L have no common zeros, π is holomorphic. Then $L \cong \pi^*\mathcal{O}(1)$, and the curvature of L is a pullback of the Kähler form on $\mathbb{C}P^n$.

REMARK: The converse is false:
a nef bundle is not necessarily semiample.

The hyperkähler SYZ conjecture

CONJECTURE: (Tyurin, Bogomolov, Hassett-Tschinkel, Huybrechts, Sawon). Any hyperkähler manifold can be deformed to a manifold admitting a holomorphic Lagrangian fibration.

REMARK: This is the only known source of SpLag fibrations.

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = q(\eta,\eta)^n$, for some rational quadratic form q on $H^2(M)$.

DEFINITION: This form is called **Bogomolov-Beauville-Fujiki form**. It is defined uniquely, up to a sign.

A trivial observation: Let $\pi: M \longrightarrow X$ be a holomorphic Lagrangian fibration, and ω_X a Kähler class on X. **Then $\eta := \pi^*\omega_X$ is nef, and satisfies $q(\eta,\eta) = 0$.**

The hyperkähler SYZ conjecture: Let L be a nef line bundle on a hyperkähler manifold, with $q(L,L) = 0$. Then L is semiample.

Semipositive line bundles

DEFINITION: A holomorphic line bundle is called **semipositive** if it has a (smooth) metric with semipositive curvature. It is obviously nef.

MAIN THEOREM: Let L be a semipositive line bundle on a hyperkähler manifold, with $q(L,L)=0$. Then L^k is effective, for some $k>0$.

Plan of a proof:

Step 1. Show that $H^*(L^N)$ is non-zero, for all N.

Step 2. Construct an embedding

$$H^i(L^N) \hookrightarrow H^0(\Omega^{2n-i}(M)\otimes L^N).$$

Step 3. THEOREM: Let L be a nef bundle on a hyperkähler manifold, with $q(L,L)=0$. Assume that $H^0(\Omega^*(M)\otimes L^N)\neq 0$, for infinitely many values of N. **Then L^k is effective, for some $k>0$.**

Step 1. Show that $H^*(L^N)$ is non-zero, for all N.

This is actually clear, because $\chi(L) = P(q(L,L))$, where P is a polynomial with coefficients depending on Chern classes of M only (Fujiki). Then

$$\chi(L) = \chi(\mathcal{O}_M) = n+1$$

(Bochner's vanishing).

Step 2. Construct an embedding

$$H^i(L^N \otimes K) \hookrightarrow H^0(\Omega^{2n-i}(M) \otimes L^N).$$

This is called **"Hard Lefschetz theorem with coefficients in L"** (Takegoshi, Mourougane, Demailly-Peternell-Schneider).

Idea of a proof: Let $B := L^*$. Then

$$\Delta_{\nabla'} - \Delta_{\overline{\partial}} = [\Theta_B, \Lambda] \leqslant 0,$$

therefore $H^i(B^N) = \ker \Delta_{\overline{\partial}} \subset \ker \Delta_{\nabla'}$, and the last space is identified with B^*-valued holomorphic differential forms.

If L has a semi-positive singular metric, a similar map exists, with coefficients in appropriate multiplier ideals.

Kobayashi-Hitchin correspondence

DEFINITION: Let F be a coherent sheaf over an n-dimensional compact Kähler manifold M. Let

$$\operatorname{slope}(F) := \frac{1}{\operatorname{rank}(F)} \int_M \frac{c_1(F) \wedge \omega^{n-1}}{vol(M)}.$$

A torsion-free sheaf F is called **(Mumford-Takemoto) stable** if for all subsheaves $F' \subset F$ one has $\operatorname{slope}(F') < \operatorname{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

DEFINITION: A Hermitian metric on a holomorphic vector bundle B is called **Yang-Mills** (Hermitian-Einstein) if the curvature of its Chern connection satisfies $\Theta_B \wedge \omega^{n-1} = \operatorname{slope}(F) \cdot \operatorname{Id}_B \cdot \omega^n$. A Yang-Mills connection is a Chern connection induced by the Yang-Mills metric.

REMARK: Yang-Mills connections minimize the integral

$$\int_M |\Theta_B|^2 \operatorname{Vol}_M$$

Kobayashi-Hitchin correspondence (part 2)

Kobayashi-Hitchin correspondence (Donaldson, Uhlenbeck-Yau) Let B be a holomorphic vector bundle. **Then B admits Yang-Mills metric if and only if B is polystable.**

COROLLARY: Any tensor product of polystable bundles is polystable.

EXAMPLE: Let M be a Kähler-Einstein manifold. Then TM is polystable.

REMARK: Let M be a Calabi-Yau (e.g., hyperkähler) manifold. Then TM admits a Hermitian-Einstein metric for any Kähler class (Calabi-Yau theorem). **Therefore, TM is stable for all Kähler structures.**

Positive currents

DEFININION: A current is a differential form with coefficients in distributions (generalized functions).

REMARK: De Rham differential is well defined on the space of currents, the Poincare lemma holds, and **cohomology of currents are the same as cohomology of differential forms.**

REMARK: The space of k-currents on an n-manifold M is dual to the space of $(n-k)$-forms with compact support.

EXAMPLE: For any subvariety $Z \subset M$ of codimension k, the map $\eta \longrightarrow \int_Z \eta$ on k-forms defines a k-current, called **the current of integration**.

DEFININION: Let M be a complex n-manifold. A **positive current** is a (1,1)-current ζ which satisfies $\langle \zeta, \alpha \rangle \geqslant 0$, for any positive $(n-1, n-1)$-form with compact support.

EXAMPLE: A current of integration over a divisor is always positive.

DEFININION: A cohomology class $\theta \in H^{1,1}(M)$ is called **pseudoeffective** if it can be represented by a closed, positive current.

Subsheaves in tensor bundles have pseudoeffective $-c_1(E)$

THEOREM: Let M be a compact hyperkähler manifold, $\mathfrak{T}$ a tensor power of a tangent bundle (such as a bundle of holomorphic forms), and $E \subset \mathfrak{T}$ a coherent subsheaf of $\mathfrak{T}$. **Then the class $-c_1(E) \in H^{1,1}_{\mathbb{R}}(M)$ is pseudoeffective**.

Step 0:

$$\int_M \alpha_{-1} \wedge ... \wedge \alpha_{2n} = \frac{1}{2n!} \sum q(\alpha_{i_1}, \alpha_{i_2}) q(\alpha_{i_3}, \alpha_{i_3})...$$

where $\alpha_i \in H^2(M)$, and the sum is taken over all $2n$-tuples (Fujiki). We chose the sign of q in such a way that $q(\omega, \omega) > 0$ for any Kähler class.

Step 1: Since $\mathfrak{T}$ is polystable, slope$(E) \leqslant 0$. Then $\int_M c_1(E) \wedge \omega^{n-1} \leqslant 0$ for any Kähler class ω. Equivalently, $q(c_1(E), \omega) \leqslant 0$. This means that **the class $-c_1(E)$ lies in the dual nef cone.**

Subsheaves in tensor bundles have pseudoeffective $-c_1(E)$ (part 2)

Step 2: Let $M_\alpha \xrightarrow{\varphi} M$ be a hyperkähler manifold birationally equivalent to M. Then φ is non-singular in codimension 1. Therefore, $H^2(M) = H^2(M_\alpha)$.

Step 3: Let $\mathfrak{T}_\alpha$ be the same tensor power of TM_α as $\mathfrak{T}$. Clearly, $\mathfrak{T}_\alpha$ can be obtained as a saturation of $\varphi^*\mathfrak{T}$. Taking a saturation of $\varphi^*E \subset \varphi^*\mathfrak{T}$, we obtain a coherent subsheaf $E_\alpha \subset \mathfrak{T}_\alpha$, with $c_1(E_\alpha) = c_1(E)$.

Step 4: We obtained that **the class $-c_1(E)$ lies in the dual nef cone of M_α, for all birational models of M.**

Step 5: We call the union of nef cones for all birational hyperkähler models of M **the birational nef cone**. The birational nef cone is dual to the pseudoeffective cone (Huybrechts, Boucksom). **Therefore, $-c_1(E)$ is pseudoeffective.**

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L-valued holomorphic forms are non-singular in codimension 1

LEMMA: Hodge's index theorem. Let $L \in H^{1,1}(M)$ a nef class satisfying $q(L,L) = 0$, and $\nu_0 \in H^{1,1}(M)$ a class satisfying $q(L,\nu_0) = 0$ and $q(\nu_0,\nu_0) \geqslant 0$. **Then L is proportional to ν_0.**

THEOREM: Let M be a compact hyperkähler manifold, L a nef line bundle satisfying $q(L,L) = 0$, $\mathfrak{T}$ some tensor power of a tangent bundle, and $\gamma \in H^0(\mathfrak{T} \otimes L)$. **Assume no power of L is effective. Then γ is non-singular in codimension 1.**

Step 1: Let L_0 be a rank 1 subsheaf of $\mathfrak{T}$ generated by $\gamma \otimes L^{-1}$. **Then $\nu := -c_1(L_0)$ is pseudoeffective.**

Step 2: By definition, γ is a section of a rank one sheaf $L \otimes L_0$. Therefore, $D = c_1(L \otimes L_0)$, where D is a union of all divisorial components of the zero set of γ. We have $c_1(L) = D + \nu$.

L-valued holomorphic forms are non-singular in codimension 1 (part 2)

Step 3: We have $c_1(L) = D + \nu$. Since L is nef, ν and D are pseudoeffective, we have $q(L, \nu) \geqslant 0$ and $q(L, D) \geqslant 0$. Then

$$0 = q(L, L) = q(L, \nu) + q(L, D) \geqslant 0.$$

We obtain that $q(L, \nu) = q(L, D) = 0$.

Step 4: Divisorial Zariski decomposition (Boucksom).
For any pseudoeffective ν, we have $\nu = \nu_0 + \sum \alpha_i E_i$, where ν_0 is birational nef, α_i positive and rational, and E_i are exceptional divisors.

Step 5: The same argument as in Step 3 can be used to show that $q(L, \nu_0) = q(L, E_i) = 0$.

Step 6: Hodge index theorem implies $\nu_0 = \lambda c_1(L)$. This gives

$$c_1(L) = D + \lambda c_1(L) + \sum \alpha_i E_i$$

Therefore, $(1 - \lambda)c_1(L)$ is effective. By our assumptions, L is not effective. **Therefore, $\lambda - 1 = 0$, and $D + \sum \alpha_i E_i = 0$.**

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From L-valued differential forms to sections of L

THEOREM: Let L be a nef bundle on a hyperkähler manifold, with $q(L,L) = 0$. Assume that $H^0(\Omega^*(M)\otimes L^N)\neq 0$, for infinitely many values of N. **Then L^k is effective, for some $k>0$.**

Step 1: Suppose that $L^{\otimes k}$ is never effective. **Then any non-zero section of $\Omega^*(M)\otimes L^N$ is non-degenerate outside of codimension 2**, as we have just shown.

Step 2: Let $E_k\subset\bigoplus_i \Omega^i M$ be subsheaf generated by global sections of $E\otimes L^{\otimes i}$, $i=1,...,k$. Let $E_\infty := \bigcup E_k$, and r be its rank. For any r-tuple of linearly independent (at generic point) sections of E_∞, $\gamma_1\in E\otimes L^{\otimes i_1}, ..., \gamma_r\in E\otimes L^{\otimes i_r}$, the determinant $\gamma_1\wedge ...\wedge\gamma_r$ is a section of $\det E_\infty\otimes L^N$, $N=\sum i_k$. non-vanishing in codimension 1, hence non-degenerate.

Step 3: This gives an isomorphism $\det E_\infty \cong L^N$**,** with $N=\sum i_k$ as above.

Step 4: There are infinitely many choices of γ_i, with i_k going to ∞, hence $\det E_\infty\cong L^N$ cannot always hold. **Contadiction! We proved that L^k is effective.**

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Multiplier ideal sheaves.

REMARK: If L is nef, it does not imply that L is semipositive. However, **a singular semipositive metric always exists.**

THEOREM (*): Let M be a simple hyperkähler manifold, L a nef bundle on M, with positive singular metric, $q(L,L)=0$, and let $\mathcal{I}(L^m)$ be the sheaf of L^2-integrable holomorphic sections of L^m. **Assume that for infinitely many $m>0$, $H^i(\mathcal{I}(L^m))\neq 0$. Then L^N is effective, for some $N>0$.**

Proof: Using the multiplier ideal version of hard Lefschetz, we obtain that $H^*(\mathcal{I}(L^m))\neq 0$ implies that $H^0(\Omega^*(M)\otimes L^m)$ is non-zero. Applying the above theorem, we obtain that L^k is effective. ■

SPECULATION: Let L be a singular nef bundle. Consider a function $k \xrightarrow{\chi_L} \chi(\mathcal{I}(L^k))$. Is it possible that $\chi_L(0)=n+1$, and $\chi_L(k)=0$ for all $k>0$, except a finite number?

If it is impossible, assumptions of (*) hold, and L^N is effective.

REMARK: **If L has algebraic singularities, $\chi_L(k)$ is either periodic, or unbounded, hence L^N is effective.**