# Hyperkähler SYZ conjecture

Misha Verbitsky

**December 18, 2008,**

**Havana, Cuba**

## Complex manifolds

**DEFINITION:** Let $M$ be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

**DEFINITION:** An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{0,1}M$. In this case $I$ is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

**THEOREM:** (Newlander-Nirenberg)
**This definition is equivalent to the usual one.**

## Kähler manifolds

**DEFINITION:** An Riemannian metric $g$ on an almost complex manifiold $M$ is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

**DEFINITION:** The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of $(M, I, g)$.

**THEOREM:** Let $(M, I, g)$ be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure $I$ is integrable, and the Hermitian form $\omega$ is closed.

(ii) One has $\nabla(I) = 0$, where $\nabla$ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

**DEFINITION:** A complex Hermitian manifold $M$ is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form $\omega$ is called **the Kähler class** of $M$. The set of all Kähler classes is called **the Kähler cone**.

## Hyperkähler manifolds

**DEFINITION:** A **hypercomplex manifold** is a manifold $M$ equipped with three complex structure operators $I, J, K$, satisfying **quaternionic relations**

$$IJ = -JI = K, \quad I^2 = J^2 = K^2 = -\mathrm{Id}_{TM}$$

(the last equation is a part of the definition of almost complex structures).

**DEFINITION:** A **hyperkähler manifold** is a hypercomplex manifold equipped with a metric $g$ which is Kähler with respect to $I, J, K$.

**REMARK:** This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves $I, J, K$.

**DEFINITION:** Let $M$ be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of $M$.

**REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$** (the group of all endomorphisms preserving $I, J, K$).

## Holomorphically symplectic manifolds

**DEFINITION:** A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

**DEFINITION:** For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

**DEFINITION:** A hyperkähler manifold $M$ is called **simple** if $H^1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

**Bogomolov's decomposition:** Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

**Remark:** A simple hyperkähler manifold is always simply connected (Cheeger-Gromoll theorem).

**Further on, all hyperkähler manifolds are assumed to be simple**.

## Holomorphic Lagrangian fibrations

**THEOREM:** (Matsushita, 1997)
Let $\pi: M \longrightarrow X$ be a surjective holomorphic map from a hyperkähler manifold $M$ to $X$, whith $0 < \dim X < \dim M$. **Then $\dim X = 1/2 \dim M$, and the fibers of $\pi$ are holomorphic Lagrangian** (this means that the symplectic form vanishes on $\pi^{-1}(x)$).

**DEFINITION:** Such a map is called **holomorphic Lagrangian fibration**.

**REMARK:** The base of $\pi$ is conjectured to be rational. Hwang (2007) proved that $X \cong \mathbb{C}P^n$, if it is smooth. Matsushita (2000) proved that it has the same rational cohomology as $\mathbb{C}P^n$.

**REMARK:** The base of $\pi$ has a natural flat connection on the smooth locus of $\pi$. The combinatorics of this connection can be used to determine the topology of $M$ (Strominger-Yau-Zaslow, Kontsevich-Soibelman).

**If we want to learn something about $M$, it's recommended to start from a holomorphic Lagrangian fibration (if it exists).**

## The SYZ conjecture

**DEFINITION:** Let $(M,\omega)$ be a Calabi-Yau manifold, $\Omega$ the holomorphic volume form, and $Z \subset M$ a real analytic subvariety, Lagrangian with respect to $\omega$. If $\Omega\Big|_Z$ is proportional to the Riemannian volume form, $Z$ is called **special Lagrangian** (SpLag).

(Harvey-Lawson): **SpLag subvarieties minimize Riemannian volume in their cohomology class.** This implies that their moduli are finite-dimensional.

**A trivial remark:** A holomorphic Lagrangian subvariety of a hyperkähler manifold $(M,I)$ is special Lagrangian on $(M,J)$, where $(I,J,K)$ is a quaternionic structure associated with the hyperkähler structure.

**Another trivial remark:** A smooth fiber of a Lagrangian fibration has trivial tangent bundle. In particular, **a smooth fiber of a holomorphic Lagrangian fibration is a torus.**

**Strominger-Yau-Zaslow, "Mirror symmetry as T-duality" (1997).** Any Calabi-Yau manifold admits a Lagrangian fibration with special Lagrangian fibers. Taking its dual fibration, one obtains "the mirror dual" Calabi-Yau manifold.

## Ample bundles

**REMARK:** Let $L$ be a holomorphic line bundle. For any metric on $L$ one associates its Chern connection; the curvature $\Theta$ of this connection is a closed, imaginary $(1,1)$-form. If the form $-\sqrt{-1}\,\Theta$ is Kähler, $L$ is called **positive**.

**REMARK:** This is the usual source of Kähler metrics in complex geometry.

**REMARK: The form $-\sqrt{-1}\,\Theta$ is Kähler if and only if $\Theta(x,\overline{x}) > 0$ for any non-zero $x \in T^{0,1}(M)$.**

**DEFINITION:** A holomorphic line bundle is called **ample**, if for a sufficiently big $N$, the tensor power $L^N$ is generated by global holomorphic sections, without common zeros, and, moreover, the natural map $M \longrightarrow \mathbb{P}H^0(L^N)^*$ is an embedding.

**THEOREM:** (Kodaira) **Let $L$ be line bundle on a compact complex manifold, with $c_1(L)$ Kähler. Then $L$ is ample.**

## Nef classes and semiample bundles

**DEFINITION:** A cohomology class $\theta$ is called **nef** (numerically effective) if it belongs to the closure of the Kähler cone. A holomorphic line bundle $L$ is **nef** if $c_1(L)$ is nef.

**DEFINITION:** A line bundle is called **semiample** if $L^N$ is generated by its holomorphic sections, which have no common zeros.

**REMARK: From semiampleness it obviously follows that $L$ is nef.** Indeed, let $\pi: M \longrightarrow \mathbb{P}H^0(L^N)^*$ the the standard map. Since sections of $L$ have no common zeros, $\pi$ is holomorphic. Then $L \cong \pi^*\mathcal{O}(1)$, and the curvature of $L$ is a pullback of the Kähler form on $\mathbb{C}P^n$.

**REMARK:** The converse is false:
**a nef bundle is not necessarily semiample.**

## The hyperkähler SYZ conjecture

**CONJECTURE:** (Tyurin, Bogomolov, Hassett-Tschinkel, Huybrechts, Sawon). Any hyperkähler manifold can be deformed to a manifold admitting a holomorphic Lagrangian fibration.

**REMARK: This is the only known source of SpLag fibrations.**

**THEOREM:** (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where $M$ is hyperkähler. Then $\int_M \eta^{2n} = q(\eta, \eta)^n$, for some rational quadratic form $q$ on $H^2(M)$.

**DEFINITION:** This form is called **Bogomolov-Beauville-Fujiki form**. It is defined uniquely, up to a sign.

**A trivial observation:** Let $\pi : M \longrightarrow X$ be a holomorphic Lagrangian fibration, and $\omega_X$ a Kähler class on $X$. **Then $\eta := \pi^* \omega_X$ is nef, and satisfies $q(\eta, \eta) = 0$.**

**The hyperkähler SYZ conjecture: Let $L$ be a nef line bundle on a hyperkähler manifold, with $q(L, L) = 0$. Then $L$ is semiample**.

## Semipositive line bundles

**DEFINITION:** A holomorphic line bundle is called **semipositive** if it has a (smooth) metric with semipositive curvature. It is obviously nef.

**MAIN THEOREM: Let $L$ be a semipositive line bundle on a hyperkähler manifold, with $q(L,L)=0$. Then $L^k$ is effective,** for some $k>0$.

**Plan of a proof:**

**Step 1. Show that $H^*(L^N)$ is non-zero**, for all $N$.

**Step 2. Construct an embedding**

$$H^i(L^N) \hookrightarrow H^0(\Omega^{2n-i}(M) \otimes L^N).$$

**Step 3. THEOREM:** Let $L$ be a nef bundle on a hyperkähler manifold, with $q(L,L)=0$. Assume that $H^0(\Omega^*(M)\otimes L^N)\neq 0$, for infinitely many values of $N$. **Then $L^k$ is effective, for some $k>0$.**

**Step 1. Show that $H^*(L^N)$ is non-zero, for all $N$.**

This is actually clear, because $\chi(L) = P(q(L,L))$, where $P$ is a polynomial with coefficients depending on Chern classes of $M$ only (Fujiki). Then

$$\chi(L) = \chi(\mathcal{O}_M) = n+1$$

(Bochner's vanishing).

**Step 2. Construct an embedding**

$$H^i(L^N \otimes K) \hookrightarrow H^0(\Omega^{2n-i}(M) \otimes L^N).$$

This is called **"Hard Lefschetz theorem with coefficients in $L$"** (Takegoshi, Mourougane, Demailly-Peternell-Schneider).

**Idea of a proof:** Let $B := L^*$. Then

$$\Delta_{\nabla'} - \Delta_{\overline{\partial}} = [\Theta_B, \Lambda] \leqslant 0,$$

therefore $H^i(B^N) = \ker \Delta_{\overline{\partial}} \subset \ker \Delta_{\nabla'}$, and the last space is identified with $B^*$-valued holomorphic differential forms.

**If $L$ has a semi-positive singular metric, a similar map exists, with coefficients in appropriate multiplier ideals.**

## Kobayashi-Hitchin correspondence

**DEFINITION:** Let $F$ be a coherent sheaf over an $n$-dimensional compact Kähler manifold $M$. Let

$$\operatorname{slope}(F) := \frac{1}{\operatorname{rank}(F)} \int_M \frac{c_1(F) \wedge \omega^{n-1}}{vol(M)}.$$

A torsion-free sheaf $F$ is called **(Mumford-Takemoto) stable** if for all subsheaves $F' \subset F$ one has $\operatorname{slope}(F') < \operatorname{slope}(F)$. If $F$ is a direct sum of stable sheaves of the same slope, $F$ is called **polystable**.

**DEFINITION:** A Hermitian metric on a holomorphic vector bundle $B$ is called **Yang-Mills** (Hermitian-Einstein) if the curvature of its Chern connection satisfies $\Theta_B \wedge \omega^{n-1} = \operatorname{slope}(F) \cdot \operatorname{Id}_B \cdot \omega^n$. A Yang-Mills connection is a Chern connection induced by the Yang-Mills metric.

**REMARK: Yang-Mills connections minimize the integral**

$$\int_M |\Theta_B|^2 \operatorname{Vol}_M$$

## Kobayashi-Hitchin correspondence (part 2)

**Kobayashi-Hitchin correspondence** (Donaldson, Uhlenbeck-Yau) Let $B$ be a holomorphic vector bundle. **Then $B$ admits Yang-Mills metric if and only if $B$ is polystable.**

**COROLLARY:** Any tensor product of polystable bundles is polystable.

**EXAMPLE:** Let $M$ be a Kähler-Einstein manifold. Then $TM$ is polystable.

**REMARK:** Let $M$ be a Calabi-Yau (e.g., hyperkähler) manifold. Then $TM$ admits a Hermitian-Einstein metric for any Kähler class (Calabi-Yau theorem). **Therefore, $TM$ is stable for all Kähler structures.**

## Positive currents

**DEFININION: A current** is a differential form with coefficients in distributions (generalized functions).

**REMARK:** De Rham differential is well defined on the space of currents, the Poincare lemma holds, and **cohomology of currents are the same as cohomology of differential forms.**

**REMARK:** The space of $k$-currents on an $n$-manifold $M$ is dual to the space of $(n-k)$-forms with compact support.

**EXAMPLE:** For any subvariety $Z \subset M$ of codimension $k$, the map $\eta \longrightarrow \int_Z \eta$ on $k$-forms defines a $k$-current, called **the current of integration**.

**DEFININION:** Let $M$ be a complex $n$-manifold. A **positive current** is a (1,1)-current $\zeta$ which satisfies $\langle \zeta, \alpha \rangle \geqslant 0$, for any positive $(n-1, n-1)$-form with compact support.

**EXAMPLE: A current of integration over a divisor is always positive.**

**DEFININION:** A cohomology class $\theta \in H^{1,1}(M)$ is called **pseudoeffective** if it can be represented by a closed, positive current.

## Subsheaves in tensor bundles have pseudoeffective $-c_1(E)$

**THEOREM:** Let $M$ be a compact hyperkähler manifold, $\mathfrak{T}$ a tensor power of a tangent bundle (such as a bundle of holomorphic forms), and $E \subset \mathfrak{T}$ a coherent subsheaf of $\mathfrak{T}$. **Then the class $-c_1(E) \in H^{1,1}_{\mathbb{R}}(M)$ is pseudoeffective**.

**Step 0:**

$$\int_M \alpha_{-1} \wedge ... \wedge \alpha_{2n} = \frac{1}{2n!} \sum q(\alpha_{i_1}, \alpha_{i_2}) q(\alpha_{i_3}, \alpha_{i_3})...$$

where $\alpha_i \in H^2(M)$, and the sum is taken over all $2n$-tuples (Fujiki). We chose the sign of $q$ in such a way that $q(\omega, \omega) > 0$ for any Kähler class.

**Step 1:** Since $\mathfrak{T}$ is polystable, slope($E$) $\leqslant 0$. Then $\int_M c_1(E) \wedge \omega^{n-1} \leqslant 0$ for any Kähler class $\omega$. Equivalently, $q(c_1(E), \omega) \leqslant 0$. This means that **the class $-c_1(E)$ lies in the dual nef cone.**

## Subsheaves in tensor bundles have pseudoeffective $-c_1(E)$ (part 2)

**Step 2:** Let $M_\alpha \xrightarrow{\varphi} M$ be a hyperkähler manifold birationally equivalent to $M$. Then $\varphi$ is non-singular in codimension 1. Therefore, $H^2(M) = H^2(M_\alpha)$.

**Step 3:** Let $\mathfrak{T}_\alpha$ be the same tensor power of $TM_\alpha$ as $\mathfrak{T}$. Clearly, $\mathfrak{T}_\alpha$ can be obtained as a saturation of $\varphi^*\mathfrak{T}$. Taking a saturation of $\varphi^* E \subset \varphi^*\mathfrak{T}$, we obtain a coherent subsheaf $E_\alpha \subset \mathfrak{T}_\alpha$, with $c_1(E_\alpha) = c_1(E)$.

**Step 4:** We obtained that **the class $-c_1(E)$ lies in the dual nef cone of $M_\alpha$, for all birational models of $M$.**

**Step 5:** We call the union of nef cones for all birational hyperkähler models of $M$ **the birational nef cone**. The birational nef cone is dual to the pseudoeffective cone (Huybrechts, Boucksom). **Therefore, $-c_1(E)$ is pseudoeffective.**

■

## $L$-valued holomorphic forms are non-singular in codimension 1

**LEMMA:** Hodge's index theorem. Let $L \in H^{1,1}(M)$ a nef class satisfying $q(L,L)=0$, and $\nu_0 \in H^{1,1}(M)$ a class satisfying $q(L,\nu_0)=0$ and $q(\nu_0,\nu_0)\geqslant 0$. **Then $L$ is proportional to $\nu_0$.**

**THEOREM:** Let $M$ be a compact hyperkähler manifold, $L$ a nef line bundle satisfying $q(L,L)=0$, $\mathfrak{T}$ some tensor power of a tangent bundle, and $\gamma \in H^0(\mathfrak{T}\otimes L)$. **Assume no power of $L$ is effective. Then $\gamma$ is non-singular in codimension 1.**

**Step 1:** Let $L_0$ be a rank 1 subsheaf of $\mathfrak{T}$ generated by $\gamma\otimes L^{-1}$. **Then $\nu := -c_1(L_0)$ is pseudoeffective.**

**Step 2: By definition, $\gamma$ is a section of a rank one sheaf $L\otimes L_0$.** Therefore, $D = c_1(L\otimes L_0)$, where $D$ is a union of all divisorial components of the zero set of $\gamma$. We have $c_1(L) = D+\nu$.

## $L$-valued holomorphic forms are non-singular in codimension 1 (part 2)

**Step 3:** We have $c_1(L) = D + \nu$. Since $L$ is nef, $\nu$ and $D$ are pseudoeffective, we have $q(L, \nu) \geqslant 0$ and $q(L, D) \geqslant 0$. Then

$$0 = q(L, L) = q(L, \nu) + q(L, D) \geqslant 0.$$

**We obtain that $q(L, \nu) = q(L, D) = 0$.**

**Step 4: Divisorial Zariski decomposition (Boucksom).**
For any pseudoeffective $\nu$, we have $\nu = \nu_0 + \sum \alpha_i E_i$, where $\nu_0$ is birational nef, $\alpha_i$ positive and rational, and $E_i$ are exceptional divisors.

**Step 5:** The same argument as in Step 3 can be used to show that $q(L, \nu_0) = q(L, E_i) = 0$.

**Step 6:** Hodge index theorem implies $\nu_0 = \lambda c_1(L)$. This gives

$$c_1(L) = D + \lambda c_1(L) + \sum \alpha_i E_i$$

Therefore, $(1 - \lambda)c_1(L)$ is effective. By our assumptions, $L$ is not effective. **Therefore, $\lambda - 1 = 0$, and $D + \sum \alpha_i E_i = 0$.**

■

## From $L$-valued differential forms to sections of $L$

**THEOREM:** Let $L$ be a nef bundle on a hyperkähler manifold, with $q(L,L)=0$. Assume that $H^0(\Omega^*(M)\otimes L^N)\neq 0$, for infinitely many values of $N$. **Then $L^k$ is effective, for some $k>0$.**

**Step 1:** Suppose that $L^{\otimes k}$ is never effective. **Then any non-zero section of $\Omega^*(M)\otimes L^N$ is non-degenerate outside of codimension 2**, as we have just shown.

**Step 2:** Let $E_k\subset\bigoplus_i\Omega^iM$ be subsheaf generated by global sections of $E\otimes L^{\otimes i}$, $i=1,...,k$. Let $E_\infty:=\bigcup E_k$, and $r$ be its rank. For any $r$-tuple of linearly independent (at generic point) sections of $E_\infty$, $\gamma_1\in E\otimes L^{\otimes i_1},...,\gamma_r\in E\otimes L^{\otimes i_r}$, the determinant $\gamma_1\wedge...\wedge\gamma_r$ is a section of $\det E_\infty\otimes L^N$, $N=\sum i_k$. non-vanishing in codimension 1, hence non-degenerate.

**Step 3: This gives an isomorphism $\det E_\infty\cong L^N$,** with $N=\sum i_k$ as above.

**Step 4:** There are infinitely many choices of $\gamma_i$, with $i_k$ going to $\infty$, hence $\det E_\infty\cong L^N$ cannot always hold. **Contadiction! We proved that $L^k$ is effective.**

■

## Multiplier ideal sheaves.

**REMARK:** If $L$ is nef, it does not imply that $L$ is semipositive. However, **a singular semipositive metric always exists.**

**THEOREM (*):** Let $M$ be a simple hyperkähler manifold, $L$ a nef bundle on $M$, with positive singular metric, $q(L,L)=0$, and let $\mathcal{I}(L^m)$ be the sheaf of $L^2$-integrable holomorphic sections of $L^m$. **Assume that for infinitely many $m>0$, $H^i(\mathcal{I}(L^m))\neq 0$. Then $L^N$ is effective, for some $N>0$.**

**Proof:** Using the multiplier ideal version of hard Lefschetz, we obtain that $H^*(\mathcal{I}(L^m))\neq 0$ implies that $H^0(\Omega^*(M)\otimes L^m)$ is non-zero. Applying the above theorem, we obtain that $L^k$ is effective. ■

**SPECULATION:** Let $L$ be a singular nef bundle. Consider a function $k \xrightarrow{\chi_L} \chi(\mathcal{I}(L^k))$. Is it possible that $\chi_L(0)=n+1$, and $\chi_L(k)=0$ for all $k>0$, except a finite number?

**If it is impossible, assumptions of (*) hold, and $L^N$ is effective.**

**REMARK:** **If $L$ has algebraic singularities, $\chi_L(k)$ is either periodic, or unbounded, hence $L^N$ is effective.**