Hypermähler SYZ conjecture

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Complex manifolds

**DEFINITION:** Let $M$ be a smooth manifold. An **almost complex structure** is an operator $I : TM \rightarrow TM$ which satisfies $I^2 = -\text{Id}_{TM}$.

The eigenvalues of this operator are $\pm \sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

**DEFINITION:** An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{0,1}M$. In this case $I$ is called a **complex structure operator**. A manifold with an integrable almost complex structure is called a **complex manifold**.

**THEOREM:** (Newlander-Nirenberg)
This definition is equivalent to the usual one.
Kähler manifolds

**DEFINITION:** An Riemannian metric \( g \) on an almost complex manifold \( M \) is called **Hermitian** if \( g(Ix, Iy) = g(x, y) \). In this case, \( g(x, Iy) = g(Ix, I^2y) = -g(y, Ix) \), hence \( \omega(x, y) := g(x, Iy) \) is skew-symmetric.

**DEFINITION:** The differential form \( \omega \in \Lambda^{1,1}(M) \) is called the **Hermitian form** of \((M, I, g)\).

**THEOREM:** Let \((M, I, g)\) be an almost complex Hermitian manifold. Then the following conditions are equivalent.

(i) The complex structure \( I \) is integrable, and the Hermitian form \( \omega \) is closed.

(ii) One has \( \nabla(I) = 0 \), where \( \nabla \) is the Levi-Civita connection

\[
\nabla : \text{End}(TM) \rightarrow \text{End}(TM) \otimes \Lambda^1(M).
\]

**DEFINITION:** A complex Hermitian manifold \( M \) is called **Kähler** if either of these conditions hold. The cohomology class \([\omega] \in H^2(M)\) of a form \( \omega \) is called the **Kähler class** of \( M \). The set of all Kähler classes is called the **Kähler cone**.
Hyperkähler manifolds

**DEFINITION:** A hypercomplex manifold is a manifold $M$ equipped with three complex structure operators $I, J, K$, satisfying quaternionic relations

$$IJ = -JI = K, \quad I^2 = J^2 = K^2 = -\text{Id}_{TM}$$

(the last equation is a part of the definition of almost complex structures).

**DEFINITION:** A hyperkähler manifold is a hypercomplex manifold equipped with a metric $g$ which is Kähler with respect to $I, J, K$.

**REMARK:** This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves $I, J, K$.

**DEFINITION:** Let $M$ be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called the holonomy group of $M$.

**REMARK:** A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving $I, J, K$).
**Holomorphically symplectic manifolds**

**DEFINITION:** A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

**DEFINITION:** For the rest of this talk, a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.

**DEFINITION:** A hyperkähler manifold $M$ is called **simple** if $H^1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

**Bogomolov's decomposition:** Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

**Remark:** A simple hyperkähler manifold is always simply connected (Cheeger-Gromoll theorem).

**Further on, all hyperkähler manifolds are assumed to be simple.**
Holomorphic Lagrangian fibrations

**THEOREM:** (Matsushita, 1997)
Let \( \pi : M \to X \) be a surjective holomorphic map from a hyperkähler manifold \( M \) to \( X \), with \( 0 < \dim X < \dim M \). Then \( \dim X = 1/2 \dim M \), and the fibers of \( \pi \) are holomorphic Lagrangian (this means that the symplectic form vanishes on \( \pi^{-1}(x) \)).

**DEFINITION:** Such a map is called holomorphic Lagrangian fibration.

**REMARK:** The base of \( \pi \) is conjectured to be rational. Hwang (2007) proved that \( X \cong \mathbb{C}P^n \), if it is smooth. Matsushita (2000) proved that it has the same rational cohomology as \( \mathbb{C}P^n \).

**REMARK:** The base of \( \pi \) has a natural flat connection on the smooth locus of \( \pi \). The combinatorics of this connection can be used to determine the topology of \( M \) (Strominger-Yau-Zaslow, Kontsevich-Soibelman).

If we want to learn something about \( M \), it’s recommended to start from a holomorphic Lagrangian fibration (if it exists).
The SYZ conjecture

**DEFINITION:** Let \((M, \omega)\) be a Calabi-Yau manifold, \(\Omega\) the holomorphic volume form, and \(Z \subset M\) a real analytic subvariety, Lagrangian with respect to \(\omega\). If \(\Omega\big|_Z\) is proportional to the Riemannian volume form, \(Z\) is called **special Lagrangian** (SpLag).

(Harvey-Lawson): **SpLag subvarieties minimize Riemannian volume in their cohomology class.** This implies that their moduli are finite-dimensional.

**A trivial remark:** A holomorphic Lagrangian subvariety of a hyperkähler manifold \((M, I)\) is special Lagrangian on \((M, J)\), where \((I, J, K)\) is a quaternionic structure associated with the hyperkähler structure.

**Another trivial remark:** A smooth fiber of a Lagrangian fibration has trivial tangent bundle. In particular, **a smooth fiber of a holomorphic Lagrangian fibration is a torus.**

Ample bundles

**REMARK:** Let $L$ be a holomorphic line bundle. For any metric on $L$ one associates its Chern connection; the curvature $\Theta$ of this connection is a closed, imaginary $(1,1)$-form. If the form $-\sqrt{-1}\Theta$ is Kähler, $L$ is called **positive**.

**REMARK:** This is the usual source of Kähler metrics in complex geometry.

**REMARK:** The form $-\sqrt{-1}\Theta$ is Kähler if and only if $\Theta(x,\overline{x}) > 0$ for any non-zero $x \in T^{0,1}(M)$.

**DEFINITION:** A holomorphic line bundle is called **ample**, if for a sufficiently big $N$, the tensor power $L^N$ is generated by global holomorphic sections, without common zeros, and, moreover, the natural map $M \to \mathbb{P}H^0(L^N)^*$ is an embedding.

**THEOREM:** (Kodaira) Let $L$ be line bundle on a compact complex manifold, with $c_1(L)$ Kähler. Then $L$ is ample.
Nef classes and semiample bundles

**DEFINITION:** A cohomology class $\theta$ is called nef (numerically effective) if it belongs to the closure of the Kähler cone. A holomorphic line bundle $L$ is nef if $c_1(L)$ is nef.

**DEFINITION:** A line bundle is called semiample if $L^N$ is generated by its holomorphic sections, which have no common zeros.

**REMARK:** From semiampleness it obviously follows that $L$ is nef. Indeed, let $\pi: M \to \mathbb{P}H^0(L^N)^*$ the the standard map. Since sections of $L$ have no common zeros, $\pi$ is holomorphic. Then $L \cong \pi^* \mathcal{O}(1)$, and the curvature of $L$ is a pullback of the Kähler form on $\mathbb{C}P^n$.

**REMARK:** The converse is false: a nef bundle is not necessarily semiample.
The hyperkähler SYZ conjecture

**CONJECTURE:** (Tyurin, Bogomolov, Hassett-Tschinkel, Huybrechts, Sawon). Any hyperkähler manifold can be deformed to a manifold admitting a holomorphic Lagrangian fibration.

**REMARK:** This is the only known source of SpLag fibrations.

**THEOREM:** (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where $M$ is hyperkähler. Then $\int_M \eta^{2n} = q(\eta, \eta)^n$, for some rational quadratic form $q$ on $H^2(M)$.

**DEFINITION:** This form is called *Bogomolov-Beauville-Fujiki form*. It is defined uniquely, up to a sign.

**A trivial observation:** Let $\pi : M \rightarrow X$ be a holomorphic Lagrangian fibration, and $\omega_X$ a Kähler class on $X$. Then $\eta := \pi^* \omega_X$ is nef, and satisfies $q(\eta, \eta) = 0$.

**The hyperkähler SYZ conjecture:** Let $L$ be a nef line bundle on a hyperkähler manifold, with $q(L, L) = 0$. Then $L$ is semiample.
Semipositive line bundles

**DEFINITION:** A holomorphic line bundle is called **semipositive** if it has a (smooth) metric with semipositive curvature. It is obviously nef.

**MAIN THEOREM:** Let $L$ be a semipositive line bundle on a hyperkähler manifold, with $q(L, L) = 0$. Then $L^k$ is effective, for some $k > 0$.

**Plan of a proof:**

**Step 1.** Show that $H^*(L^N)$ is non-zero, for all $N$.

**Step 2.** Construct an embedding

$$H^i(L^N) \hookrightarrow H^0(\Omega^{2n-i}(M) \otimes L^N).$$

**Step 3.** **THEOREM:** Let $L$ be a nef bundle on a hyperkähler manifold, with $q(L, L) = 0$. Assume that $H^0(\Omega^*(M) \otimes L^N) \neq 0$, for infinitely many values of $N$. Then $L^k$ is effective, for some $k > 0$.  

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Step 1. Show that $H^*(L^N)$ is non-zero, for all $N$.

This is actually clear, because $\chi(L) = P(q(L, L))$, where $P$ is a polynomial with coefficients depending on Chern classes of $M$ only (Fujiki). Then

$$\chi(L) = \chi(O_M) = n + 1$$

(Bochner's vanishing).

Step 2. Construct an embedding

$$H^i(L^N \otimes K) \hookrightarrow H^0(\Omega^{2n-i}(M) \otimes L^N).$$

This is called “Hard Lefschetz theorem with coefficients in $L$” (Takegoshi, Mourougane, Demailly-Peternell-Schneider).

Idea of a proof: Let $B := L^*$. Then

$$\Delta_{\nabla'} - \Delta_{\overline{\partial}} = [\Theta_B, \wedge] \leq 0,$$

therefore $H^i(B^N) = \ker \Delta_{\overline{\partial}} \subset \ker \Delta_{\nabla'}$, and the last space is identified with $B^*$-valued holomorphic differential forms.

If $L$ has a semi-positive singular metric, a similar map exists, with coefficients in appropriate multiplier ideals.
Kobayashi-Hitchin correspondence

**DEFINITION:** Let $F$ be a coherent sheaf over an $n$-dimensional compact Kähler manifold $M$. Let

$$\text{slope}(F) := \frac{1}{\text{rank}(F)} \int_M \frac{c_1(F) \wedge \omega^{n-1}}{\text{vol}(M)}.$$  

A torsion-free sheaf $F$ is called *(Mumford-Takemoto) stable* if for all sub-sheaves $F' \subset F$ one has $\text{slope}(F') < \text{slope}(F)$. If $F$ is a direct sum of stable sheaves of the same slope, $F$ is called **polystable**.

**DEFINITION:** A Hermitian metric on a holomorphic vector bundle $B$ is called *(Yang-Mills) (Hermitian-Einstein)* if the curvature of its Chern connection satisfies $\Theta_B \wedge \omega^{n-1} = \text{slope}(F) \cdot \text{Id}_B \cdot \omega^n$. A Yang-Mills connection is a Chern connection induced by the Yang-Mills metric.

**REMARK:** Yang-Mills connections minimize the integral

$$\int_M |\Theta_B|^2 \text{Vol}_M$$
Kobayashi-Hitchin correspondence (part 2)

Kobayashi-Hitchin correspondence (Donaldson, Uhlenbeck-Yau) Let $B$ be a holomorphic vector bundle. Then $B$ admits Yang-Mills metric if and only if $B$ is polystable.

**COROLLARY:** Any tensor product of polystable bundles is polystable.

**EXAMPLE:** Let $M$ be a Kähler-Einstein manifold. Then $TM$ is polystable.

**REMARK:** Let $M$ be a Calabi-Yau (e.g., hyperkähler) manifold. Then $TM$ admits a Hermitian-Einstein metric for any Kähler class (Calabi-Yau theorem). Therefore, $TM$ is stable for all Kähler structures.
Positive currents

**DEFINITION:** A current is a differential form with coefficients in distributions (generalized functions).

**REMARK:** De Rham differential is well defined on the space of currents, the Poincare lemma holds, and cohomology of currents are the same as cohomology of differential forms.

**REMARK:** The space of $k$-currents on an $n$-manifold $M$ is dual to the space of $(n-k)$-forms with compact support.

**EXAMPLE:** For any subvariety $Z \subset M$ of codimension $k$, the map $\eta \mapsto \int_Z \eta$ on $k$-forms defines a $k$-current, called the current of integration.

**DEFINITION:** Let $M$ be a complex $n$-manifold. A positive current is a $(1,1)$-current $\zeta$ which satisfies $\langle \zeta, \alpha \rangle \geq 0$, for any positive $(n-1,n-1)$-form with compact support.

**EXAMPLE:** A current of integration over a divisor is always positive.

**DEFINITION:** A cohomology class $\theta \in H^{1,1}(M)$ is called pseudoeffective if it can be represented by a closed, positive current.
**Subsheaves in tensor bundles have pseudoeffective** $-c_1(E)$

**THEOREM:** Let $M$ be a compact hyperkähler manifold, $\mathcal{I}$ a tensor power of a tangent bundle (such as a bundle of holomorphic forms), and $E \subset \mathcal{I}$ a coherent subsheaf of $\mathcal{I}$. **Then the class** $-c_1(E) \in H^{1,1}_\mathbb{R}(M)$ **is pseudoeffective.**

**Step 0:**

$$\int_M \alpha_1 \wedge \ldots \wedge \alpha_{2n} = \frac{1}{2n!} \sum q(\alpha_{i_1}, \alpha_{i_2}) q(\alpha_{i_3}, \alpha_{i_3}) \ldots$$

where $\alpha_i \in H^2(M)$, and the sum is taken over all $2n$-tuples (Fujiki). We chose the sign of $q$ in such a way that $q(\omega, \omega) > 0$ for any Kähler class.

**Step 1:** Since $\mathcal{I}$ is polystable, $\text{slope}(E) \leq 0$. Then $\int_M c_1(E) \wedge \omega^{n-1} \leq 0$ for any Kähler class $\omega$. Equivalently, $q(c_1(E), \omega) \leq 0$. This means that **the class** $-c_1(E)$ **lies in the dual nef cone.**
Subsheaves in tensor bundles have pseudoeffective $-c_1(E)$ (part 2)

**Step 2:** Let $M_\alpha \xrightarrow{\varphi} M$ be a hyperkähler manifold birationally equivalent to $M$. Then $\varphi$ is non-singular in codimension 1. Therefore, $H^2(M) = H^2(M_\alpha)$.

**Step 3:** Let $\mathfrak{T}_\alpha$ be the same tensor power of $TM_\alpha$ as $\mathfrak{T}$. Clearly, $\mathfrak{T}_\alpha$ can be obtained as a saturation of $\varphi^*\mathfrak{T}$. Taking a saturation of $\varphi^*E \subset \varphi^*\mathfrak{T}$, we obtain a coherent subsheaf $E_\alpha \subset \mathfrak{T}_\alpha$, with $c_1(E_\alpha) = c_1(E)$.

**Step 4:** We obtained that the class $-c_1(E)$ lies in the dual nef cone of $M_\alpha$, for all birational models of $M$.

**Step 5:** We call the union of nef cones for all birational hyperkähler models of $M$ the birational nef cone. The birational nef cone is dual to the pseudoeffective cone (Huybrechts, Boucksom). Therefore, $-c_1(E)$ is pseudoeffective.
**Lemma:** Hodge’s index theorem. Let $L \in H^{1,1}(M)$ a nef class satisfying $q(L, L) = 0$, and $\nu_0 \in H^{1,1}(M)$ a class satisfying $q(L, \nu_0) = 0$ and $q(\nu_0, \nu_0) \geq 0$. Then $L$ is proportional to $\nu_0$.

**Theorem:** Let $M$ be a compact hyperkähler manifold, $L$ a nef line bundle satisfying $q(L, L) = 0$, $\mathcal{I}$ some tensor power of a tangent bundle, and $\gamma \in H^0(\mathcal{I} \otimes L)$. Assume no power of $L$ is effective. Then $\gamma$ is non-singular in codimension 1.

**Step 1:** Let $\mathcal{L}_0$ be a rank 1 subsheaf of $\mathcal{I}$ generated by $\gamma \otimes L^{-1}$. Then $\nu := -c_1(\mathcal{L}_0)$ is pseudoeffective.

**Step 2:** By definition, $\gamma$ is a section of a rank one sheaf $L \otimes \mathcal{L}_0$. Therefore, $D = c_1(L \otimes \mathcal{L}_0)$, where $D$ is a union of all divisorial components of the zero set of $\gamma$. We have $c_1(L) = D + \nu$. 

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**L-valued holomorphic forms are non-singular in codimension 1 (part 2)**

**Step 3:** We have $c_1(L) = D + \nu$. Since $L$ is nef, $\nu$ and $D$ are pseudoeffective, we have $q(L, \nu) \geq 0$ and $q(L, D) \geq 0$. Then

$$0 = q(L, L) = q(L, \nu) + q(L, D) \geq 0.$$  

We obtain that $q(L, \nu) = q(L, D) = 0$.

**Step 4:** **Divisorial Zariski decomposition (Boucksom).**

For any pseudoeffective $\nu$, we have $\nu = \nu_0 + \sum \alpha_i E_i$, where $\nu_0$ is birational nef, $\alpha_i$ positive and rational, and $E_i$ are exceptional divisors.

**Step 5:** The same argument as in Step 3 can be used to show that $q(L, \nu_0) = q(L, E_i) = 0$.

**Step 6:** Hodge index theorem implies $\nu_0 = \lambda c_1(L)$. This gives

$$c_1(L) = D + \lambda c_1(L) + \sum \alpha_i E_i$$

Therefore, $(1 - \lambda)c_1(L)$ is effective. By our assumptions, $L$ is not effective. **Therefore, $\lambda - 1 = 0$, and $D + \sum \alpha_i E_i = 0$.**
From \( L \)-valued differential forms to sections of \( L \)

**THEOREM:** Let \( L \) be a nef bundle on a hyperkähler manifold, with \( q(L, L) = 0 \). Assume that \( H^0(\Omega^*(M) \otimes L^N) \neq 0 \), for infinitely many values of \( N \). Then \( L^k \) is effective, for some \( k > 0 \).

**Step 1:** Suppose that \( L^k \) is never effective. Then any non-zero section of \( \Omega^*(M) \otimes L^N \) is non-degenerate outside of codimension 2, as we have just shown.

**Step 2:** Let \( E_k \subset \bigoplus_i \Omega^i M \) be subsheaf generated by global sections of \( E \otimes L^i \), \( i = 1, ..., k \). Let \( E_\infty := \bigcup E_k \), and \( r \) be its rank. For any \( r \)-tuple of linearly independent (at generic point) sections of \( E_\infty \), \( \gamma_1 \in E \otimes L^i_1, ..., \gamma_r \in E \otimes L^i_r \), the determinant \( \gamma_1 \wedge ... \wedge \gamma_r \) is a section of \( \text{det} E_\infty \otimes L^N \), \( N = \sum i_k \). non-vanishing in codimension 1, hence non-degenerate.

**Step 3:** This gives an isomorphism \( \text{det} E_\infty \cong L^N \), with \( N = \sum i_k \) as above.

**Step 4:** There are infinitely many choices of \( \gamma_i \), with \( i_k \) going to \( \infty \), hence \( \text{det} E_\infty \cong L^N \) cannot always hold. **Contradiction!** We proved that \( L^k \) is effective.
Multiplier ideal sheaves.

**REMARK:** If $L$ is nef, it does not imply that $L$ is semipositive. However, a singular semipositive metric always exists.

**THEOREM (*)&:** Let $M$ be a simple hyperkähler manifold, $L$ a nef bundle on $M$, with positive singular metric, $q(L, L) = 0$, and let $\mathcal{I}(L^m)$ be the sheaf of $L^2$-integrable holomorphic sections of $L^m$. **Assume that for infinitely many** $m > 0$, $H^i(\mathcal{I}(L^m)) \neq 0$. **Then** $L^N$ is effective, for some $N > 0$.

**Proof:** Using the multiplier ideal version of hard Lefschetz, we obtain that $H^*(\mathcal{I}(L^m)) \neq 0$ implies that $H^0(\Omega^*(M) \otimes L^m)$ is non-zero. Applying the above theorem, we obtain that $L^k$ is effective. 

**SPECULATION:** Let $L$ be a singular nef bundle. Consider a function $k \xrightarrow{\chi_L} \chi(\mathcal{I}(L^k))$. Is it possible that $\chi_L(0) = n + 1$, and $\chi_L(k) = 0$ for all $k > 0$, except a finite number?

**If it is impossible, assumptions of (*) hold, and** $L^N$ is effective.

**REMARK:** If $L$ has algebraic singularities, $\chi_L(k)$ is either periodic, or unbounded, hence $L^N$ is effective.