SYZ conjecture and ergodic theory

Misha Verbitsky

Seminar of Laboratory of Algebraic Geometry,

HSE, Aug. 26, 2015

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure I is integrable, and the Hermitian form ω is closed.

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M. The set of all Kähler classes is called **the Kähler cone**.

Hyperkähler manifolds

DEFINITION: A **hypercomplex manifold** is a manifold M equipped with three complex structure operators I, J, K, satisfying **quaternionic relations**

$$IJ = -JI = K, \quad I^2 = J^2 = K^2 = -\operatorname{Id}_{TM}$$

(the last equation is a part of the definition of almost complex structures).

DEFINITION: A **hyperkähler manifold** is a hypercomplex manifold equipped with a metric g which is Kähler with respect to I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

Holomorphic Lagrangian fibrations

THEOREM: (Matsushita, 1997)
Let $\pi: M \longrightarrow X$ be a surjective holomorphic map from a hyperkähler manifold M to X, whith $0 < \dim X < \dim M$. **Then** $\dim X = 1/2 \dim M$**, and the fibers of** π **are holomorphic Lagrangian** (this means that the symplectic form vanishes on $\pi^{-1}(x)$).

DEFINITION: Such a map is called **holomorphic Lagrangian fibration**.

REMARK: The base of π is conjectured to be rational. Hwang (2007) proved that $X \cong \mathbb{C}P^n$, if it is smooth. Matsushita (2000) proved that it has the same rational cohomology as $\mathbb{C}P^n$.

REMARK: The base of π has a natural flat connection on the smooth locus of π. The combinatorics of this connection can be used to determine the topology of M (Strominger-Yau-Zaslow, Kontsevich-Soibelman).

If we want to learn something about M, it's recommended to start from a holomorphic Lagrangian fibration (if it exists).

The SYZ conjecture

DEFINITION: Let (M, ω) be a Calabi-Yau manifold, Ω the holomorphic volume form, and $Z \subset M$ a real analytic subvariety, Lagrangian with respect to ω. If $\Omega|_Z$ is proportional to the Riemannian volume form, Z is called **special Lagrangian** (SpLag).

(Harvey-Lawson): **SpLag subvarieties minimize Riemannian volume in their cohomology class.** This implies that their moduli are finite-dimensional.

A trivial remark: A holomorphic Lagrangian subvariety of a hyperkähler manifold (M, I) is special Lagrangian on (M, J), where (I, J, K) is a quaternionic structure associated with the hyperkähler structure.

Another trivial remark: A smooth fiber of a Lagrangian fibration has trivial tangent bundle. In particular, **a smooth fiber of a holomorphic Lagrangian fibration is a torus.**

Strominger-Yau-Zaslow, "Mirror symmetry as T-duality" (1997). Any Calabi-Yau manifold admits a Lagrangian fibration with special Lagrangian fibers. Taking its dual fibration, one obtains "the mirror dual" Calabi-Yau manifold.

Ample bundles

REMARK: Let L be a holomorphic line bundle. For any metric on L one associates its Chern connection; the curvature Θ of this connection is a closed, imaginary $(1,1)$-form. If the form $-\sqrt{-1}\,\Theta$ is Kähler, L is called **positive**.

REMARK: This is the usual source of Kähler metrics in complex geometry.

REMARK: The form $-\sqrt{-1}\,\Theta$ is Kähler if and only if $\Theta(x,\overline{x}) > 0$ for any non-zero $x \in T^{0,1}(M)$.

DEFINITION: A holomorphic line bundle is called **ample**, if for a sufficiently big N, the tensor power L^N is generated by global holomorphic sections, without common zeros, and, moreover, the natural map $M \longrightarrow \mathbb{P}H^0(L^N)^*$ is an embedding.

THEOREM: (Kodaira) **Let L be line bundle on a compact complex manifold, with $c_1(L)$ Kähler. Then L is ample.**

Nef classes and semiample bundles

DEFINITION: A cohomology class θ is called **nef** (numerically effective) if it belongs to the closure of the Kähler cone. A holomorphic line bundle L is **nef** if $c_1(L)$ is nef.

DEFINITION: A line bundle is called **semiample** if L^N is generated by its holomorphic sections, which have no common zeros.

REMARK: From semiampleness it obviously follows that L is nef. Indeed, let $\pi: M \longrightarrow \mathbb{P}H^0(L^N)^*$ the the standard map. Since sections of L have no common zeros, π is holomorphic. Then $L \cong \pi^*\mathcal{O}(1)$, and the curvature of L is a pullback of the Kähler form on $\mathbb{C}P^n$.

REMARK: The converse is false:
a nef bundle is not necessarily semiample.

The hyperkähler SYZ conjecture

CONJECTURE: (Tyurin, Bogomolov, Hassett-Tschinkel, Huybrechts, Sawon). Any hyperkähler manifold can be deformed to a manifold admitting a holomorphic Lagrangian fibration.

REMARK: This is the only known source of SpLag fibrations.

A trivial observation: Let $\pi : M \longrightarrow X$ be a holomorphic Lagrangian fibration, and ω_X a Kähler class on X. **Then $\eta := \pi^* \omega_X$ is nef, and satisfies $q(\eta, \eta) = 0$.**

The hyperkähler SYZ conjecture: Let L be a nef line bundle on a hyperkähler manifold, with $\int_M c_1(L)^{\dim_{\mathbb{C}} M} = 0$ (such bundle is called **parabolic**). **Then L is semiample**.

REMARK: This statement is a special case of Kawamata's abundance conjecture. **When $\int_M c_1(L)^{\dim_{\mathbb{C}} M} > 0$, Kawamata's base point free theorem implies that L is semiample.**

SYZ conjecture in dimension 4, $\mathbb{Q}$-effectivity in any dimension

Two applications of today's lecture main theorem (stated later):

THEOREM: Let M be a hyperkähler manifold, and L a parabolic nef bundle. Then L is **$\mathbb{Q}$-effective**, that is, its **positive power $L^{\otimes N}$ admits a holomorphic section.**

THEOREM: (based on a theorem Gongyo and Matsumura)
Let M be a hyperkähler manifold, $\dim_{\mathbb{C}} M = 4$. **Then SYZ conjecture is true.**

Currents and generalized functions

DEFINITION: Let F be a Hermitian bundle with connection ∇, on a Riemannian manifold M with Levi-Civita connection, and

$$\|f\|_{C^k} := \sup_{x \in M} \left(|f| + |\nabla f| + ... + |\nabla^k f| \right)$$

the corresponding **C^k-norm** defined on smooth sections with compact support. **The C^k-topology is independent from the choice of connection and metrics.**

DEFINITION: A generalized function is a functional on top forms with compact support, which is continuous in one of C^i-topologies.

DEFINITION: A k-current is a functional on $(\dim M - k)$-forms with compact support, which is continuous in one of C^i-topologies.

REMARK: Currents are forms with coefficients in generalized functions.

Currents on complex manifolds

DEFINITION: The space of currents is equipped with **weak topology** (a sequence of currents converges if it converges on all forms with compact support). The space of currents with this topology is a **Montel space** (barrelled, locally convex, all bounded subsets are precompact). Montel spaces are **reflexive** (the map to its double dual with strong topology is an isomorphism).

CLAIM: De Rham differential is continuous on currents, and the Poincare lemma holds. Hence, **the cohomology of currents are the same as cohomology of smooth forms.**

DEFINITION: On an complex manifold, **(p,q)-currents** are (p,q)-forms with coefficients in generalized functions

REMARK: In the literature, this is sometimes called $(n-p,n-q)$-currents.

CLAIM: The Dolbeault lemma holds on (p,q)-currents, and **the $\overline{\partial}$-cohomology are the same as for forms.**

Positive forms and currents

DEFINITION: A **weakly positive (p,p)-form** onis a real (p,p)-form η which satisfies $\eta(x_1, Ix_1, x_2, Ix_2, ...x_p, Ix_p) \geqslant 0$ for all $x_1, ...n_p \in TM$. **The set of weakly positive (p,p)-forms is a convex cone.**

DEFINITION: A cone of strongly positive (p,p)-forms is a convex cone generated by $\eta_1 \wedge \eta_2 \wedge ... \wedge \eta_p$, for all poisitve (1,1)-forms $\eta_1, ..., \eta_p$.

CLAIM: For $(n-1, n-1)$-forms, strong positivity is the same as weak.

CLAIM: The cones of strongly and weakly positive forms are dual.

REMARK: The 0 form is weakly positive and strongly positive.

DEFINITION: A **strongly/weakly positive (p,p)-current** is a current taking non-negative values on weakly/strongly positive compactly supported $(n-p, n-p)$-forms.

REMARK: A positive (p,p)-current is C^0-continuous.

Positive currents and measures

DEFINITION: A **positive generalized function** is a generalized function taking non-negative values on all positive volume forms.

REMARK: Positive generalized functions are C^0-continuous. A positive generalized function multiplied by a positive volume form **gives a measure on a manifold,** and all measures are obtained this way.

DEFINITION: A **mass** of a positive (p,p)-current η on a Hermitian n-manifold (M,ω) is a measure $\eta\wedge\omega^{n-p}$. **It is non-negative, and positive, if $\eta\neq 0$.**

Theorem: The space of positive currents with bounded mass is (weakly) compact.

Proof: Follows from precompactness of bounded sets in weak-*-topology.

REMARK: Since the space of currents is Montel, **all bounded subsets are precompact.**

Closed positive currents and psh functions

DEFINITION: Let $Z \subset M$ be a complex analytic subvariety. **The current of integration** $[Z]$ is the current $\alpha \longrightarrow \int_Z \alpha$. **It is closed and positive** (Lelong).

REMARK: (Poincare-Lelong formula) $\frac{\sqrt{-1}}{\pi} dd^c \log|\varphi| = [Z_\varphi]$, where Z_φ is a divisor of a holomorphic function φ.

DEFINITION: A locally integrable function $f: M \longrightarrow [\infty, \infty[$ is called **plurisubharmonic** (psh) if $dd^c f$ is a positive current.

CLAIM: (a local dd^c-lemma) **Locally, every positive, closed (1,1)-current is obtained as $dd^c f$,** for some psh function f.

DEFINITION: Let f be a real locally integrable function on a complex manifold, such that $dd^c f + \alpha$ is a positive current, for some smooth $(1,1)$-form α. Then f is called **almost plurisubharmonic**.

DEFINITION: Let L be a line bundle and h a smooth Hermitian metric on L. For any almost plurisubharmonic function f, we call he^{-f} **a singular metric** on L. Its curvature is equal to $\Theta_h + dd^c f$.

Nef current

DEFINITION: A nef current is a limit of positive, closed (p,p)-forms in the space of currents.

REMARK: Generally speaking, **currents cannot be multiplied.** However, if a current is nef, $\eta = \lim \eta_i$, where η_i are positive, closed forms, we can define $\eta\wedge\Theta$ for any positive current as a limit $\lim_i \eta_i\wedge\Theta$. This limit exists by compactness, it is closed and positive, but not necessarily unique.

THEOREM: Let $x\in M$ be a point on a Hermitian manifold, and $\eta_x := dd^c \log d_x$, where $d_x(y) := d(x,y)$ is the distance function. Then in a sufficiently small neighbourhood of x, the current η_x is a positive, closed, nef current, obtained as a limit $\eta_x = \lim_i \eta_i$, where $\eta_i = dd^c(\max(-i, \log d_x)$. Moreover, **the limit $\lim \eta_i\wedge\Theta$ is uniquely defined for any positive, closed current Θ.**

Lelong numbers

DEFINITION: Let Θ be a closed, positive (p,p)-current on an n-manifold M, $x \in M$ a point, $\eta_x = dd^c \log d_x$, and $\Theta \wedge \eta_x^{n-p}$ the corresponding measure. Consider its Lebesgue decomposition $\Theta \wedge \eta_x^{n-p} = c\delta_x + \alpha$, where $\alpha(x) = 0$ and δ_x is an atomic measure concentrated in x. The number c is called the **Lelong number** of a current Θ is x, denoted by $\nu_x(\Theta)$.

DEFINITION: Lelong set F_c of a current Θ is a set of all x where Θ has non-zero Lelong number $\nu_x(\Theta) \geqslant c$.

THEOREM: (Siu) Lelong sets of a positive, closed current **are closed and complex analytic for any $c \in \mathbb{R}^{\geqslant 0}$.**

Lelong numbers and multiplier ideals

DEFINITION: Let f be an almost plurisubharmonic function, and e^{-f} the corresponding singular metric on a trivial line bundle $\mathcal{O}_M$. **The multiplier ideal** of f is a sheaf of L^2-integrable holomorphic sections of $\mathcal{O}_M$.

THEOREM: (Nadel) It is a coherent sheaf, equal to $\mathcal{O}_M$ outside of the set of all points where $\nu_x(dd^c f) \leqslant 1/n$.

REMARK: The multiplier ideal of f is determined uniquely by the corresponding current $dd^c f$.

REMARK: $e^{-2\varphi}$ **is integrable in** x **if and only if the multiplier ideal of** φ **is trivial in** x**.**

Lelong numbers and SYZ conjecture

THEOREM: (V., 2009) Let L be a parabolic bundle on a hyperkähler manifold. Assume that L admits a metric with all Lelong numbers vanishing. **Then L is $\mathbb{Q}$-effective.**

THEOREM: (Gongyo-Matsumura) Let L be a parabolic bundle on a hyperkähler manifold M, $\dim_{\mathbb{C}} M = 4$. Assume that L admits a metric with all Lelong numbers vanishing. **Then L is semiample.**

The main results of today's lecture are the following two theorems:

THEOREM: Let M be a hyperkähler manifold, and $[\eta] \in H^{1,1}(M)$ a parabolic nef class. **Then $[\eta]$ is represented by a positive, closed (1,1)-current with vanishing Lelong numbers.**

REMARK: This implies $\mathbb{Q}$-effectivity and SYZ conjecture for $\dim_{\mathbb{C}} = 4$.

Uniqueness of positive representative

DEFINITION: An automorphism of a hyperkähler manifold is called **hyperbolic** if it acts on $H^2(M)$ with eigenvalue α, where $|\alpha| > 1$.

THEOREM: Let (M, I) be a hyperkähler manifold with Pic(M) non-maximal. Assume that a deformation of M admits a hyperbolic automorphism. Consider an irrational parabolic nef class $[\eta] \in H^{1,1}(M)$, and let η be a positive, closed (1,1)-current representing $[\eta]$. **Then such a current is unique in its cohomology class.**

Teichmüller spaces

DEFINITION: Let M be a smooth manifold. **A complex structure** on M is an endomorphism $I \in \operatorname{End} TM$, $I^2 = -\operatorname{Id}_{TM}$ such that the eigenspace bundles of I are **involutive**, that is, satisfy satisfy $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$.

REMARK: Let Comp be the space of such tensors equipped with a topology of convergence of all derivatives. **It is a Fréchet manifold**.

REMARK: The diffeomorphism group Diff is a Fréchet Lie group acting on a Fréchet manifold Comp in a natural way.

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Let $\operatorname{Teich} := \operatorname{Comp} / \operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Computation of the mapping class group

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$,** and $c > 0$ a rational number.

DEFINITION: The form q is called **Bogomolov-Beauville-Fujiki form**. It has signature $(3, b_2 - 3)$.

THEOREM: (V., 1996, 2009) Let M be a simple hyperkähler manifold, and $\Gamma_0 = \operatorname{Aut}(H^*(M,\mathbb{Z}), p_1, ..., p_n)$. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}), q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}), q)$ **has finite kernel.**

The period map

REMARK: To simplify the language, we redefine Teich and Comp for hyperkähler manifolds, admitting only complex structures of Kähler type. Since the Hodge numbers are constant in families of Kähler manifolds, **for any $J \in$ Teich, (M,J) is also a simple hyperkähler manifold, hence $H^{2,0}(M,J)$ is one-dimensional.**

Definition: Let Per : Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map J to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map Per : Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called **the period map**.

REMARK: Per **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}\mathrm{er} := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \ | \ \ q(l,l) = 0, q(l,\bar{l}) > 0\}.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}\mathrm{er} = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$**.** Indeed, the group $SO(H^2(M,\mathbb{R}),q) = SO(b_2-3,3)$ acts transitively on $\mathbb{P}\mathrm{er}$, and $SO(2)\times SO(b_2-3,1)$ is a stabilizer of a point.

Birational Teichmüller moduli space

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (Huybrechts, 2001) Two points $I, I' \in \operatorname{Teich}$ **are non-separable if and only if there exists a bimeromorphism $(M,I) \longrightarrow (M,I')$ which is non-singular in codimension 2 and acts as identity on $H^2(M)$.**

REMARK: This is possible only if (M,I) and (M,I') contain a rational curve. **General hyperkähler manifold has no curves;** ones which have belong to a countable union of divisors in Teich.

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ is an isomorphism, for each connected component of Teich_b.

Ergodic complex structures

DEFINITION: Let (M,μ) be a space with measure, and G a group acting on M preserving measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M')=0$ or $\mu(M\backslash M')=0$.

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.** ■

DEFINITION: A complex structure $I \in$ Teich is called **ergodic** if its orbit is dense in its connected component in Teich.

CLAIM: Let (M,I) be a manifold with an ergodic complex structure, and I' its deformation. **Then there exists a sequence of diffeomorphisms ν_i such that $\nu_i(I)$ converges to I' in C^∞-topology.** Moreover, this property is equivalent to ergodicity of I.

THEOREM: Let M be a compact torus, $\dim_{\mathbb{C}} M \geqslant 2$, or a simple hyperkähler manifold. **A complex structure on M is ergodic if and only if** Pic(M) **is not of maximal rank.**

Ergodic action of Diff_0 on parabolic classes

DEFINITION: Let Teich_p be the Teichmüller space of pairs (I, η), where η is a parabolic class and $I \in \operatorname{Teich}$ a complex structure on a hyperkähler manifold

REMARK: Let $W = H^2(M, \mathbb{R})$. The space Teich_p by global Torelli is mapped (bijectively, outside of a measure 0 set) to the set $\mathbb{P}\mathrm{er}_p$ of pairs $\mathbb{P}\mathrm{er}_p = \{(u,l) \ \mid \ l \in \operatorname{Gr}_{++}(W), u \in l^\perp, q(u,u) = 0\}$ such that $\langle l, u\rangle$ has positive orientation (u should belong to a positive half of the light cone of W).

Theorem 1: The mapping class group acts on Teich_p ergodically. Moreover, any orbit of (M, η) where η is irrational and $\operatorname{Pic}(M, I)$ is not of maximal rank is **dense in $\mathbb{P}\mathrm{er}$.**

The ergodicity follows from Moore theorem.

Ergodic action of Diff_0 on parabolic classes and Moore's theorem

THEOREM: (Calvin C. Moore, 1966) Let Γ be an arithmetic lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact subgroup. Then the left action of Γ on G/H is **ergodic,** that is, **for all Γ-invariant measurable subsets $Z \subset G/H$, either Z has measure 0, or $G/H \backslash Z$ has measure 0.**

In our case, Teich_p is identified (up to measure 0) with the quotient $\mathbb{P}\mathrm{er}_p = SO(W)/H$, where $W = H^2(M, \mathbb{R})$ and H is a stabilizer of $(l, u) \in \mathbb{P}\mathrm{er}_p$, which is a non-compact parabolic group.

Ratner theory

To determine which orbits of Γ on Teich_p are dense, we use Ratner theory.

DEFINITION: Let G be a connected Lie group equipped with a Haar measure. **A lattice** $\Gamma \subset G$ is a discrete subgroup of finite covolume (that is, G/Γ has finite volume).

EXAMPLE: By Borel and Harish-Chandra theorem, **any integer lattice in a simple Lie group has finite covolume.**

THEOREM: (Marina Ratner)
Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ a lattice. Then **a closure of any H-orbit in G/Γ is an orbit of a closed, connected subgroup $S \subset G$, such that $S \cap \Gamma \subset S$ is a lattice.**

REMARK: Let $x \in G/H$ be a point in a homogeneous space, and $\Gamma \cdot x$ its Γ-orbit, where Γ is an arithmetic lattice. Then its closure is an orbit of a group S containing stabilizer of x. Moreover, **S is a smallest group defined over rationals and stabilizing x.**

Ratner theory and parabolic classes

CLAIM: Let W be a vector space with a quadratic form of signature $(3, k)$, $G = SO^+(W)$, and $H = \mathrm{St}(l, u)$ a stabilizer of an oriented pair (l, u), where $l \in \mathrm{Gr}_{++}(W)$, a u a non-zero null-vector in $l^\perp$ Then **any closed connected Lie subgroup $S \subset G$ containing H coincides with G, H, $\mathrm{St}(l)$ or $\mathrm{St}(u)$.**

COROLLARY: Let $\Gamma \subset SO^+(W)$ be an arithmetic lattice acting on G/H, and $(l, u) \in G/H$ any point. Then **any Γ-orbit of (l, u) is dense, unless l or u is rational.**

Now the second claim of Theorem 1 follows immediately.

Theorem 1: The mapping class group acts on Teich_p ergodically. Moreover, any orbit of (M, η) where η is irrational and $\mathrm{Pic}(M, I)$ is not of maximal rank is **dense in $\mathbb{P}\mathrm{er}$.**

Vanishing of Lelong numbers

THEOREM: Let M be a hyperkähler manifold, and $[\eta] \in H^{1,1}(M)$ a parabolic nef class. **Then $[\eta]$ is represented by a positive, closed (1,1)-current with vanishing Lelong numbers.**

Proof. Step 1: When $\operatorname{Pic}(M, I) = 0$, it is known. Indeed, in this case all complex subvarieties of (M, I) are symplectic, and Lelong sets are known to be coisotropic (V., 2010).

Step 2: Let (I, η) be any point in Teich_p, and $(J, [\mu]) \in \operatorname{Teich}_p$ a general point satisfying assumptions of Step 1. By Theorem 1, it has dense Γ-orbit. Then **there exists a sequence of diffeomorphisms ν_i such that $\lim_i \nu_i(J, [\mu]) = (I, [\eta])$.**

Step 3: Let μ be a positive, closed current representing $[\mu]$ on (M, J). Then its Lelong numbers vanish. The limit $\lim_i \nu_i(\mu)$ exists by compactness; it is a positive, closed current on (M, I), denoted by μ. **To prove theorem it suffices to show that its Lelong numbers vanish.**

Step 4: Denote by (J_i, η_i) the point $\nu_i(J, [\mu])$ in Teich_p, and by μ_i the current $\nu_i(\mu)$. Let β be a smooth, closed $(1,1)$-form representing $[\mu]$ on (M, J), and $\alpha_i = \nu_i(\beta)$ a sequence smooth forms representing $[\eta_i]$ on (M, J_i) and converging to a smooth form α on (M, I). Then $\mu_i - \alpha = dd^c\psi_i$ by dd^c-lemma, and $\mu - \alpha = dd^c\psi$, with ψ_i converging to ψ. By Nadel's theorem, **to prove that Lelong numbers of μ vanish it suffices to show that $e^{-C\psi}$ is L^2-integrable for any $C > 0$.**

Step 5: A Calabi-Yau manifold has a canonical measure, therefore, all diffeomorphisms ν_i are measure-preserving. With respect to this measure Vol_i on (M, J_i), the integral $\int e^{-C\psi_i} \operatorname{Vol}_i$ stays constant. Then the limit $e^{-C\psi}$ is L^2-integrable, as the following lemma implies (weak limits of positive currents $dd^c\varphi_i$ correspond to pointwise limits of plurisubharmonic functions φ_i). ■

LEMMA: Let f be a bounded from above function on a compact manifold (M, ρ) with measure, ν_i a sequence of measure-preserving self-maps, and $g = \lim \nu_i^* f$ a pointwise limit. **Then $\int_M f\rho = \int_M g\rho$.**

Proof: Let $f_C := \max(f, -C)$ and $g_C := \max(g, -C)$. Then $g_C = \lim \nu_i^* f_C$, giving $\int_M f_C\rho = \int_M g_C\rho$, by Lebesgue bounded convergence theorem. On the other hand, $\int_M g\rho = \lim_C \int_M g_C\rho$ by Lebesgue monotonous convergence theorem. ■

Uniqueness of positive representative: Dinh-Sibony result

THEOREM: (Dinh-Sibony)
Let (M,I) be a compact complex manifold equipped with a automorphism T acting on $H^{1,1}(M)$ with eigenvalues α_i, and let α be a real eigenvalue satisfying $|\alpha| > 1$. Assume that it is a maximal eigenvalue, and the rest are smaller. **Then the corresponding eigenvector $v \in H^{1,1}(M)$ is nef, and is represented by a unique positive current.**

I will prove this theorem for volume-preserving automorphisms.

Proof. Step 1: Since α is the biggest eigenvalue, and the Kähler cone is open in $H^{1,1}$, there exists a Kähler class ω which does not lie in the sum W of all other eigenspaces: $\omega = cv + w$, where $w \in W$, $c \neq 0$. **This means that the term $c\alpha^n v$ dominates the rest in $T^*\omega = c\alpha^n v + (T^n)^* w$, and we have** $\lim_i \frac{(T^n)^*\omega}{\alpha^n} = cv$**.** Therefore cv is nef.

Step 2: It remains to prove uniqueness of the positive representative η of v. Suppose that there are two positive representatives η_1, η_2, with $\eta_1 - \eta_2 = dd^c\psi$ by dd^c-lemma. Then $T^*\psi = \alpha\psi + C$, where C is a real constant. Let Vol be a volume form on M preserved by T, and consider the pushforward ψ_*Vol as a measure on $\mathbb{R}$. Then ψ_*Vol is mapped to itself by $x \longrightarrow \alpha x + C$. This is impossible, however, because such a measure must be atomic, and ψ is non-constant. ■

Uniqueness of positive representative

THEOREM: Let (M, I) be a hyperkähler manifold with Pic(M) non-maximal. Assume that a deformation of M admits a hyperbolic automorphism. Consider an irrational parabolic nef class $[\mu] \in H^{1,1}(M)$, and let μ be a positive, closed (1,1)-current representing $[\mu]$. **Then such a current is unique in its cohomology class.**

Proof: Suppose that such a current is not unique; let μ, λ be positive representatives. Let (M, J, η) be a hyperkähler manifold admitting a hyperbolic automorphism, $[\eta]$ its eigenvector with a maximal eigenvalue, and η its positive, closed representative. Using ergodic theory as above, we may approximate $(J, [\eta])$ by a sequence $\nu_i(I, [\mu])$. Then the limits $\lim_i \nu_i(\mu)$ and $\lim_i \nu_i(\lambda)$ are positive currents which represent $[\eta]$, giving $\lim_i \nu_i(\mu) = \lim_i \nu_i(\lambda)$. However, $\mu - \lambda = dd^c\psi$, and $\lim \nu_i(\psi) \neq 0$ by the same argument with Lebesgue monotone convergency as above (we again use that η_i is volume-preserving).

This implies that a positive representative is unique for any pair $(I, [\mu])$ with dense orbit. ■