Salem numbers, Siegel disks, and automorphisms of K3 surfaces

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Estruturas geométricas em variedades,

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Salem numbers

arXiv:1408.0195 Chris Smyth, *Survey article: Seventy years of Salem numbers*.

DEFINITION: A Salem number is a real algebraic number $\lambda > 1$ which is Galois conjugate to λ^{-1}, such that the rest of its conjugates satisfy $|\lambda_i| = 1$.

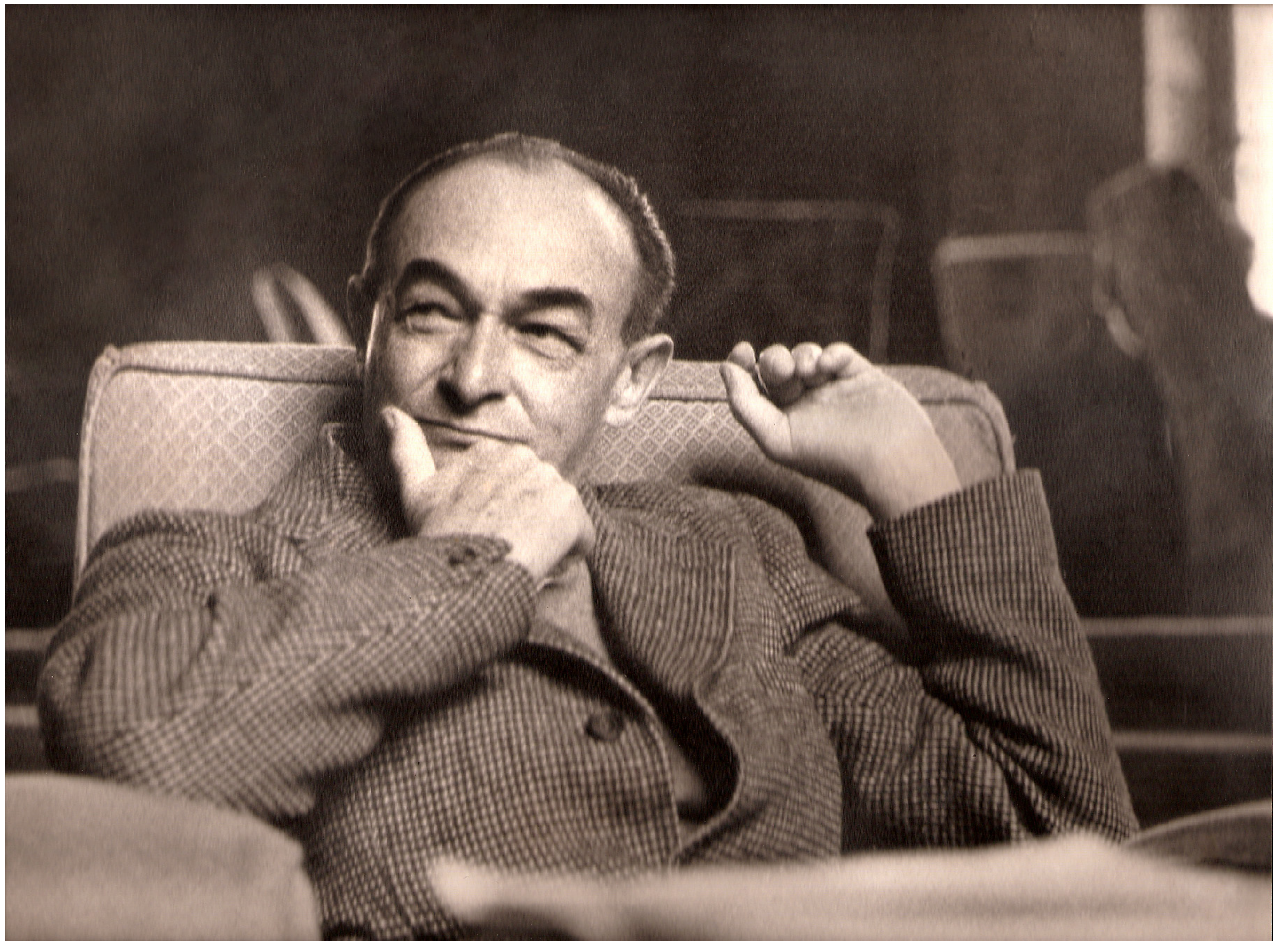
REMARK: Since λ and λ^{-1} have the same minimal polynomial $P(t) = t^n + a_{n-1}t^{n-1} + ... + a_0 \in \mathbb{Z}[t]$, this polynomial is **palindromic:** $a_i = a_{n-i}$, and $a_0 = 1$.

LEMMA: (Salem)
Let $\lambda > 1$ be a real algebraic number such that all its conjugates belong to the closed disk $|z| \leqslant 1$, with at least one on its boundary. **Then λ is a Salem number.**

Proof: Let τ be the Galois conjugate on the boundary, then $\tau^{-1} = \bar{\tau}$ is also a conjugate. **Therefore, λ is conjugate to λ^{-1}, and all other conjugates ν are conjugate to ν^{-1}.** Since both ν and ν^{-1} belong to the disk $|z| \leqslant 1$, they lie on its boundary. ■

REMARK: If λ is a Salem number, all its integer powers λ^k, $k \neq 0$, **are also Salem numbers.** Indeed, **the Galois conjugates of λ^k are powers of Galois conjugates of λ.**

Raphaël Salem (1898-1963)



Raphaël Salem (1898-1963)

Salem, R. *Algebraic numbers and Fourier analysis,* Heath mathematical monographs, 1963

Number fields containing Salem numbers

DEFINITION: An arithmetic field is a finite extension of $\mathbb{Q}$. An arithmentic field is **totally real** if the images of all its embeddings to $\mathbb{C}$ belong to $\mathbb{R}$.

THEOREM: (Salem)
A number field K is generated by a Salem number τ **if and only if it contains an index 2 totally real subfield $K_1 = \mathbb{Q}[\alpha]$, such that $\alpha > 2$ is an irrational real algebraic integer, and all its Galois conjugates belong to the interval $]-2, 2[$.**

Proof. Step 1: Let $\alpha := \tau + \tau^{-1}$. Then all Galois conjugates of α are $\nu + \overline{\nu}$, where ν is on a circle, hence they are real and belong to $]-2,2[$. We proved that $\mathbb{Q}[\tau]$ is degree 2 extension of a totally real field $K_1 = \mathbb{Q}[\alpha]$.

Step 2: Conversely, consider an index 2 totally real subfield $K_1 = \mathbb{Q}[\alpha]$ with the above properties, and let τ be the solution of the quadratic equation $\tau + \tau^{-1} = \alpha$. Then all its Galois conjugates satisfy $\tau_1 + \tau_1^{-1} = \alpha_1$, where $\alpha_1 \in]-2,2[$. The quadratic equation $\tau_1^2 - \alpha_1\tau_1 + 1 = 0$ has discriminant $\alpha_1^2 - 4 < 0$, hence it has two complex conjugate solutions; since these solutions are inverse, they lie on a circle. The number τ is a solution of $\tau^2 - \alpha\tau + 1 = 0$ which has positive discriminant $\alpha^2 - 4$ giving $\tau = \frac{\alpha + \sqrt{\alpha^2-4}}{2}$. Then $\tau > \alpha > 1$ and its inverse $\tau^{-1} = \frac{\alpha - \sqrt{\alpha^2-4}}{2}$ is its Galois conjugate. ■

Example of a Salem number

EXAMPLE: The argument of the last step **gives a way to construct explicit examples of Salem numbers.** Let $\alpha = x + \sqrt{y} > 1$ be a real quadratic irrational number such that $x - \sqrt{y} \in]-2, 2[$. Then, as follows from Step 2, the solution τ of the equation $\tau + \tau^{-1} = \alpha$ is a Salem number. One of the numbers which have this property is $\alpha = \frac{1}{2}(3 + \sqrt{5})$; since $\sqrt{5} \approx 2.23607$, $\alpha \approx 2.6$ and its Galois conjugate $\frac{1}{2}(3 - \sqrt{5}) \approx 0.4$.

Salem numbers contained in number fields

THEOREM: (Salem)
Let K be a number field generated by a Salem number τ. Then for any Salem number $\tau' \in K$, $\tau \neq \tau'$, **the fraction $\frac{\tau}{\tau'}$ is also a Salem number.**

Proof. Step 1: Let $\alpha := \tau + \tau^{-1}$. Since $\mathbb{Q}(\tau)$ is a quadratic extension of $\mathbb{Q}(\alpha)$, we can write $\tau' = p(\alpha) + \tau q(\alpha)$, for some polynomials $p, q \in \mathbb{Q}[z]$. **We are going to show that τ' has the same degree as τ.** Otherwise, some real conjugates $\tau'_i = p(\alpha_i) + \tau_i q(\alpha_i)$ would be real for τ_i non-real, implying that $q = 0$, which is impossible.

Step 2: The element τ' is expressed polynomially through τ: $\tau' = P(\tau)$, where $P \in \mathbb{Q}[z]$. Consider the Galois element ι which takes τ to τ^{-1}. This automorphism maps τ' to a real conjugate of τ', because P has rational coefficients. The only real conjugates of τ' are τ' and $(\tau')^{-1}$. **Since τ' has the same degree as τ, it cannot be fixed by ι, which implies $\iota(\tau') = (\tau')^{-1}$.**

Step 3: We obtained that $\tau\tau'$ is conjugate to its reciprocal. Since **the rest of conjugates of τ and τ' lie on the circle, the same is true for $\tau\tau'$, hence it is a Salem number.** ■

The hyperbolic space and its isometries

REMARK: The group $O(m,n), m,n>0$ has 4 connected components. We denote the connected component of 1 by $SO^+(m,n)$. We call a vector v **positive** if its square is positive.

DEFINITION: Let V be a vector space with quadratic form q of signature $(1,n)$, $\operatorname{Pos}(V) = \{x \in V \ \ | \ \ q(x,x) > 0\}$ its **positive cone**, and $\mathbb{P}^+V$ projectivization of $\operatorname{Pos}(V)$. Denote by g any $SO(V)$-invariant Riemannian structure on $\mathbb{P}^+V$. Then $(\mathbb{P}^+V, g)$ is called **hyperbolic space**, and the group $SO^+(V)$ **the group of oriented hyperbolic isometries**.

Classification of hyperbolic isometries

Theorem-definition: Let $n > 0$, and $\alpha \in SO^+(1,n)$ is a non-trivial oriented isometry acting on $V = \mathbb{R}^{1,n}$. Then one and only one of these three cases occurs

(i) α has an eigenvector x with $q(x,x) > 0$ (α is **"elliptic isometry"**)

(ii) α has an eigenvector x with $q(x,x) = 0$ and a real eigenvalue λ_x satisfying $|\lambda_x| > 1$ (α is **"hyperbolic isometry"**)

(iii) α has a unique eigenvector x with $q(x,x) = 0$ (α is **"parabolic isometry"**).

REMARK: All eigenvalues of elliptic and parabolic isometries have absolute value 1. **Hyperbolic and elliptic isometries are semisimple** (that is, diagonalizable over $\mathbb{C}$), parabolic are not.

DEFINITION: The quadric $\{l \in \mathbb{P}V \ \ | \ \ q(l,l) = 0\}$ is called **the absolute**. It is realized as the boundary of the hyperbolic space $\mathbb{P}^+V$. Then **elliptic isometries have no fixed points on the absolute, parabolic isometries have 1 fixed point on the absolute, and hyperbolic isometries have 2**.

Hyperbolic lattices

DEFINITION: A quadratic lattice (Λ, q) is $\mathbb{Z}^n$ equipped with $\mathbb{Z}$-valued quadratic form q. **An arithmetic hyperbolic lattice** is the group of $O(\Lambda, q)$ isometries of (Λ, q) of signature $(1, n-1)$.

REMARK: Clearly, $O(\Lambda, q) \subset O(1, n-1)$, hence $O(\Lambda, q)$ acts on the hyperbolic space $\mathbb{H}^{n-1}$ by isometries. It is possible to show that **the Haar measure of the quotient $\frac{O(1,n-1)}{O(\Lambda,q)}$ is finite,** and **the Riemannian volume of the quotient $\frac{\mathbb{H}^{n-1}}{O(\Lambda,q)}$ is also finite.**

DEFINITION: Let $\Gamma \subset O(1, n-1)$ be a discrete subgroup. It is called **a lattice subgroup** if the Haar measure of the quotient $\frac{O(1,n-1)}{\Gamma}$ is finite.

DEFINITION: It is not hard to see that the Riemannian volume of $\mathbb{H}^{n-1}/\Gamma$ is finite if and only if it is a lattice. In this situation, the quotient $\mathbb{H}^{n-1}/\Gamma$ is called **a hyperbolic manifold.**

Salem numbers and hyperbolic automorphisms

PROPOSITION: Let (Λ, q) be a quadratic lattice of signature $(1, n-1)$, and $O(\Lambda, q) \subset O(1, n-1)$ the corresponding arithmetic subgroup. Consider an element $u \in O(\Lambda, q)$ as an isometry of $\mathbb{H}^{n-1}$. Suppose that this isometry hyperbolic, and let $\lambda > 1$ be the corresponding eigenvalue of u. **Then λ is a Salem number.**

Proof: By definition, u can be represented by an invertible integer matrix. Let $P(t) \in \mathbb{Z}[t]$ be an irreducible factor if its minimal polynomial which satisfies $P(\lambda) = 0$. Since all roots of $P(t)$ except λ and λ^{-1} lie on a circle, λ is a Salem number. ■

Salem numbers and complex surfaces

DEFINITION: Let M be a complex surface, and $H^2(M) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$ its second cohomology with their Hodge decomposition. By Hodge index theorem, **the intersection form on $H^{1,1}(M)$ has signature $(1, m)$.** An automorphism of M is called **hyperbolic, parabolic** or **elliptic** if its action on $H^{1,1}(M)$ is hyperbolic, parabolic or elliptic.

REMARK: If the surface M is also projective, then $H^{1,1}(M)$ can be decomposed onto its integer part $H^{1,1}(M) \cap H^2(M, \mathbb{Z})$, called **the Hodge part**. The Hodge part of $H^{1,1}(M)$ has signature $(1, r)$, by projectivity. Then any hyperbolic automorphism of M acts on $H^{1,1}(M) \cap H^2(M, \mathbb{Z})$ with an eigenvalue λ which satisfies $|\lambda| > 1$; as we have shown above, $|\lambda|$ **is a Salem number.**

This is how Salem numbers crop up in McMullen's work on K3 surfaces.

McMullen, Curtis T. *Dynamics on K3 surfaces: Salem numbers and Siegel disks.* J. Reine Angew. Math. 545 (2002), 201-233.

Siegel disks

DEFINITION: A linear map $A: (z_1, z_2, ..., z_n) \longrightarrow (\lambda_1 z_1, \lambda_2 z_2, ..., \lambda_n z_n)$ is called **an irrational rotation** if $|\lambda_i| = 1$ and the action of A on $(S^1)^n$ has dense orbits. In this case the numbers λ_i are called **multiplicatively independent**.

DEFINITION: We say that a holomorphic self-map $f: M \longrightarrow M$ **admits a Siegel disk** if f has a fixed point p and a neighbourhood of p admitting coordinates where f acts linearly as an irrational rotation.

THEOREM: (McMullen)
Let M be a complex n-manifold, and $f: M \longrightarrow M$ a holomorphic map which has a fixed point p such that df acts on T_pM as an irrational rotation. Assume that all eigenvalues of this action are algebraic. **Then (M, f) admits a Siegel disk.**

The proof of this result **takes a lot of number theory,** originally developed by Gel'fond and Fel'dman in their work on Hilbert 7-th problem on transcendence of numbers such as $2^{\sqrt{2}}$.

Diophantine numbers

DEFINITION: Let $\lambda_1, ..., \lambda_n$ be non-zero complex numbers. We say that they are **multiplicatively independent** if the only solution of $\prod \lambda_i^{k_i} = 1$ is $k_1 = 0, ..., k_n = 0$. We say they are **jointly Diophantine** if there exist numbers $C, M > 0$ such that for all n-tuples $(k_1, ..., k_n) \in \mathbb{Z}^n$, we have

$$\left|\prod \lambda_i^{k_i} - 1\right| > C(\max |k_i|)^{-M}.$$

THEOREM: (S. Sternberg, 1961)
Let M be a complex n-manifold, and $f : M \longrightarrow M$ a holomorphic map which has a fixed point p such that df acts on T_pM with jointly Diophantine eigenvalues. **Then f can be linearized in a neighbourhood of p.** In other words, **(M, f) admits a Siegel disk.**

THEOREM: (N. I. Fel'dman, 1968)
Let λ_i be multiplicatively algebraic numbers. Chose their logarithms $\log \lambda_i$. **Then there exists a positive number M such that**

$$|k_0 2\pi\imath + k_1 \log \lambda_1 + ... + k_n \log \lambda_n| > e^{-M(d+\max |k_i|)},$$

where d is the degree of the field generated by λ_i.

COROLLARY: Any collection of multiplicatively independent algebraic numbers **is jointly Diophantine.**

Comparing this result and Sternberg's theorem, **we immediately obtain the result of McMullen.**

K3 surfaces

DEFINITION: A **holomorphically symplectic manifold** is a complex manifold equipped with a non-degenerate, holomorphic $(2,0)$-form.

DEFINITION: Take a 2-dimensional complex torus T, then all 16 singular points of $T/\pm 1$ are of form $\mathbb{C}^2/\pm 1$. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

DEFINITION: **A K3 surface** is a complex deformation of a Kummer surface.

CLAIM: 1. $\pi_1(K3) = 0$,
2. The second homology and cohomology of K3 is torsion-free.
3. $b_2(K3) = 22$, and the signature of its intersection form is $(3, 19)$.
4. The intersection form of K3 is even, and the corresponding quadratic lattice is $U^3 \oplus (-E_8)^2$, where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8 is the Coxeter matrix for the group E_8.

THEOREM: Any complex compact surface with $c_1(M) = 0$ and $\pi_1(M) = 0$ **is isomorphic to K3.** Moreover, **it is Kähler.**

Siegel disk on K3 surfaces

THEOREM: (McMullen)
Let M be a projective K3 surface, and f its holomorphic automorphism. **Then f cannot admit a Siegel disk.** Moreover, **the set isomorphism classes of K3 surfaces admitting an automorphism with Siegel disk is countable.**

DEFINITION: Let f be an automorphism of a K3 surface, and Ω a holomorphic symplectic form of M. Then, clearly, $f^*\Omega$ is proportional to Ω with a complex coefficient: $f^*\Omega = \delta\Omega$. Then δ is called **the determinant** of f.

REMARK: Let f be an automorphism of a K3 surface which admits a Siegel disk with eigenvalues λ_1, λ_2. Then $\lambda_1\lambda_2 = \delta$. In particular, **$\delta$ is not a root of unity.**

THEOREM: (McMullen)
Let M be a non-projective K3 surface, and f its holomorphic automorphism. Assume that $\operatorname{Tr} f^*\Big|_{H^2(M)} = -1$, and one of its eigenvalues is a Salem number. Assume, moreover, that its determinant δ satisfies $\tau := \delta + \delta^{-1} > 1 - 2\sqrt{2}$. Assume, finally, that τ has a Galois conjugate τ' such that $\tau' < 1 - 2\sqrt{2}$. **Then f has a unique fixed point p.** Moreover, **(M, f, p) admits a Siegel disk.**

Automorphisms of K3 surfaces and Salem numbers

PROPOSITION: Let f be an automorphism of a K3 surface, Then **either all eigenvalues of f^* on $H^2(M)$ are roots of unity,** or **there is a unique, simple eigenvalue λ with $|\lambda| > 1$, and $|\lambda|$ is a Salem number.**

Proof. Step 1: Let δ be the determinant of f. Any diffeomorphism of M preserves the volume: $\int_M \mathrm{Vol} = \int_M f^* \mathrm{Vol}$. Choosing $\Omega \wedge \overline{\Omega} = \mathrm{Vol}$, and using $f^* \mathrm{Vol} = \delta\bar{\delta}\, \mathrm{Vol}$, we obtain that $|\delta| = 1$. Therefore, any eigenvector of $f^*\big|_{H^2(M,\mathbb{C})}$ belongs to $H^{1,1}(M, \mathbb{R})$. From the classification of the isometries of the hyperbolic space, we obtain that **either all eigenvalues of $f^*\big|_{H^{1,1}(M)}$ satisfy $|\alpha_i| = 1$ or there exists a unique simple eigenvalue λ with $|\lambda| > 1$.**

Step 2: In the first case, we use Kronecker's theorem: **if an algebraic number α and all its conjugates lie in the unit circle, it is a root of unity.**

Step 3: In the second case, the Galois conjugates of λ are roots of the minimal polynomial of $f^*\big|_{H^2(M)}$, hence they are all eigenvalues of $f^*\big|_{H^2(M)}$, **but there are at most two eigenvalues which do not lie on the circle.** Therefore, $|\lambda|$ is a Salem number. ■

Automorphisms of projective K3 surfaces

The following theorem immediately implies that as projective K3 cannot have Siegel disks.

PROPOSITION: Let f be an automorphism of a projective K3 surface M, and δ its determinant. **Then δ is a root of unity.**

Proof: Consider the group $NS(M) := H^2(M,\mathbb{Z})\cap H^{1,1}(M)$. Since M is projective, $NS(M)$ contains a positive vector, hence the group $NS(M)\otimes_{\mathbb{Z}}\mathbb{Q}$, called **the Hodge lattice** has signature $(1,k)$, and its orthogonal complement $\mathbb{T}(M)$, called **the transcendental lattice**, has signature $(2,19-k)$. Since f^* preserves $\operatorname{Re}(H^{2,0}(M)\oplus H^{0,2}(M))\subset \mathbb{T}(M)$, which has signature $(2,0)$, it belongs to a maximal compact subgroup:

$$f^*\Big|_{\mathbb{T}(M)} \subset O(2)\times O(19-k)\subset O(\mathbb{T}(M)\otimes\mathbb{R}) = O(2,19-k).$$

However, f^* is an integer automorphism of $\mathbb{T}(M)$, hence it lies in a discrete subgroup of $O(\mathbb{T}(M)\otimes\mathbb{R})$. **Intersection of a discrete group and a compact group is always finite, hence f^* has finite order on $\mathbb{T}(M)$.** This implies that f^* acts Ω as a root of unity. ■

Hodge structures

DEFINITION: Let $V_{\mathbb{R}}$ be a real vector space. **A (real) Hodge structure of weight w** on a vector space $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is a decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called **rational Hodge structure** if one fixes a rational lattice $V_{\mathbb{Q}}$ such that $V_{\mathbb{R}} = V_{\mathbb{Q}} \otimes \mathbb{R}$, and **an integer Hodge structure** if one fixes an integer lattice $V_{\mathbb{Z}} \subset V_{\mathbb{Q}}$. A Hodge structure is equipped with $U(1)$-action, with $u \in U(1)$ acting as u^{p-q} on $V^{p,q}$. **Morphism** of Hodge structures is a rational map which is $U(1)$-invariant.

DEFINITION: A rational Hodge structure $V_{\mathbb{C}} = \bigoplus_{\substack{p+q=2 \\ p,q\geqslant 0}} V^{p,q}$ of weight 2 with $\dim V^{2,0} = 1$ is called **a Hodge structure of K3 type**.

THEOREM: (global Torelli theorem for K3)
Let M be a K3 surface, $q \in \operatorname{Sym}^2(H^2(M)^*)$ the intersection form, and S a Hodge structure $H^2(M) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$ of K3 type on $H^2(M)$. Assume that $q(l,l) = 0$ and $q(l,\bar{l}) > 0$ for a non-zero vector $l \in H^{2,0}(M)$. **Then there exists a complex structure on M inducing this Hodge decomposition. Moreover, it is unique up to a diffeomorphism acting trivially on $H^2(M,\mathbb{R})$.**

REMARK: This implies that the Teichmüller space of K3 surfaces is $\operatorname{Gr}_{++}(H^2(M,\mathbb{R}))$.

Global Torelli for K3 with automorphisms

We use the following version of Torelli theorem.

THEOREM: (global Torelli theorem for K3 with automorphisms) Let M be a K3 surface, and $\underline{f} \in O^+(H^2(M, \mathbb{Z}), q)$ an isometry of its intersection lattice preserving the orientation in the (3,0)-part. Assume that there exists a Hodge structure on $H^2(M)$ preserved by f, and satisfying $q(l, l) = 0$ and $q(l, \bar{l}) > 0$ for a non-zero vector $l \in H^{2,0}(M)$. Then **there exists a complex structure I on M inducing this Hodge decomposition, and an automorphism f of (M, I) inducing $\underline{f}$ on $H^2(M, \mathbb{R})$.** Moreover, the pair (M, I, f) **is defined uniquely** up to a diffeomorphism acting trivially on $H^2(M, \mathbb{R})$.

Only countably many K3 surfaces admit Siegel disks

THEOREM: (McMullen)
Let $H^2(M,\mathbb{Z}), q$ be an intersection lattice of a K3 surface, and f an automorphism admitting a Siegel disk. Then the action of f on $H^2(M,\mathbb{Z})$ **on the transcendental lattice has countably many Hodge structures of K3 type which are compatible with the action of f.**

Proof: The action of f on its eigenspace $H^{2,0}(M)$ has an eigenvalue which is conjugate to a Salem number, hence it is a simple eigenvalue. **Then $H^{2,0}(M)$ can be one of finitely many simple 1-dimensional eigenspaces of f,** and $H^{2,0}(M)$ determines the complex structure by Torelli theorem. ■

REMARK: This theorem **immediately implies that the set of K3 admitting Siegel disks is countable.** Indeed, the Hodge lattice is integral, hence there are only countably many choices. The transcendental lattice admits only countably many Hodge decompositions which are compatible with f by the above theorem.

Atyah-Bott fixed point formula

Let f be a holomorphic automorphism of compact Kähler n-manifold, and $L^r(f) := \sum_{s=0}^n (-1)^s \operatorname{Tr} f^*\Big|_{H^{r,s}(M)}$

THEOREM: (Atyah-Bott fixed point formula)
Assume that all fixed points of f are simple. Then

$$L^r(f) = \sum_{p_i} \frac{\operatorname{Tr}\left(Df\Big|_{\Lambda^r T_{p_i} M}\right)}{\det\left(\operatorname{Id} - D_{p_i} f\right)}.$$

where $\{p_i\}$ is the set of fixed points of f.

REMARK: Let f be an automorphism of a K3 surface with a unique fixed point p. **Then this formula gives $L^2(f) = 1 + \delta = \frac{\delta}{1 - \operatorname{Tr} D_p f + \delta}$, which can be rewritten as**

$$\operatorname{Tr} D_p f = \frac{1 + \delta + \delta^2}{1 + \delta}.$$

Existence of Siegel disks

REMARK: Existence of automorphism of the following type is implied by the Torelli theorem.

THEOREM: (McMullen)
Let M be a non-projective K3 surface, and f its holomorphic automorphism. Assume that $\operatorname{Tr} f^*\big|_{H^2(M)} = -1$, and one of its eigenvalues is a Salem number. Assume, moreover, that its determinant δ satisfies $\tau := \delta + \delta^{-1} > 1 - 2\sqrt{2}$. Assume, finally, that τ has a Galois conjugate τ' such that $\tau' < 1 - 2\sqrt{2}$. **Then f has a unique fixed point p.** Moreover, (M, f, p) **admits a Siegel disk.**

Proof. Step 1: By Lefschetz fixed point formula, f has $1 = \operatorname{Tr} f^*\big|_{H^2(M)} + 2$ fixed points, hence **the fixed point p of f is unique and simple.** It remains to find the eigenvalues of dF on T_pM.

Step 2: The eigenvalues α, β of the action of Df on T_pM are computed using the formula $\alpha + \beta = \operatorname{Tr} D_pf = \frac{1+\delta+\delta^2}{1+\delta}$. and $\alpha\beta = \det D_pf = \delta$. Computing α, β in terms of δ yields the multiplicative independence, which implies existence of a Siegel disk by application of Fel'dman's theorem. ■