Sasakian manifolds

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Contact manifolds.

In this lecture, **all manifolds are assumed to be oriented.**

Definition: Let M be a smooth manifold, dim $M = 2n-1$, and ω a symplectic form on $M \times \mathbb{R}^{>0}$. Suppose that ω is **homogeneous**: $\Psi_q^*\omega = q^2\omega$, where $\Psi_q(m,t) = (m, qt)$. Then M is called **a contact manifold**.

Remark: **The contact form** on M is defined as $\theta = \omega \lrcorner \vec{T}$, where $\vec{T} = t\frac{d}{dt}$. Then $d\theta = [d, \cdot \lrcorner \vec{T}]\omega = \operatorname{Lie}_{\vec{T}}\omega = 2\omega$. Therefore, **the form $d\theta^{n-1} \wedge \theta = \frac{1}{n}\omega^n \lrcorner \vec{T}$ is non-degenerate on $M \times \{t_0\} \subset M \times \mathbb{R}^{>0}$.**

Remark: Usually, a contact manifold is defined as a **$(2n-1)$-manifold with 1-form θ such that $d\theta^{n-1} \wedge \theta$ is nowhere degenerate.**

Example: **An odd-dimensional sphere S^{2n-1} is contact.** Indeed, $C(S^{2n-1}) = S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{R}^{2n} \backslash 0$, and symplectic form $\sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}$ is homogeneous.

Contact geometry is an odd-dimensional counterpart to symplectic geometry

Contact manifolds: three equivalent definitions

Definition 1: Let $C(S) = (S \times \mathbb{R}^{>}0)$ be a cone, equipped with the standard action $h_\lambda(x,t) = (x, \lambda t)$. Assume that $C(S)$ is equipped with a symplectic form ω such that $h_\lambda^* \omega = \lambda^2 \omega$. Then S is called **contact manifold**.

Definition 2: Let S be an odd-dimensional manifold, and $B \subset TS$ an oriented sub-bundle of codimension 1, with Frobenius form $\Lambda^2 B \xrightarrow{\Phi} TS/B$ non-degenerate. Then S is called **contact manifold**, $B \subset TS$ **the contact bundle**.

Definition 3: Let S be manifold of dimension $2k+1$, $B \subset TS$ an oriented sub-bundle of codimension 1. Assume that for a nowhere vanishing 1-form $\theta \in \Lambda^1 M$ vanishing on B, the form $\theta \wedge (d\theta)^k$ is a non-degenerate volume form. Then (S, B) is called **a contact manifold**, and θ **a contact form**.

THEOREM: **These three definitions are all equivalent.**

The proof is given later today.

Basic forms and Frobenius theorem

DEFINITION: Let M be a manifold, $B \subset TM$ a sub-bundle, $\theta \in \Lambda^i M$ a differential form. It is called **basic** with respect to B if for each $b \in B$, one has $\theta \lrcorner b = 0$ and $\operatorname{Lie}_b \theta = 0$.

DEFINITION: A sub-bundle $B \subset TM$ is called **involutive** if $[B, B] \subset B$.

THEOREM: ("Frobenius theorem")
Let $B \subset TM$ be an involutive sub-bundle. **Then for each point $x \in M$ there exists a neighbourhood $U \ni x$ and a smooth projection $\pi : U \longrightarrow N$ such that $B = \ker \pi$.** ■

THEOREM: Let M be a manifold, $B \subset TM$ an involutive sub-bundle, $\theta \in \Lambda^i M$ a differential form. **Then the following are equivalent.**

(i) η is basic.

(ii) for any open subset $U \subset M$ and a projection $\pi : U \longrightarrow N$ such that $B = \ker d\pi$, one has $\eta = \pi^* \eta'$ for some $\eta' \in \Lambda^i N$.

Contact manifolds: three equivalent definitions (proofs)

Definition 2: Let S be an odd-dimensional manifold, and $B \subset TS$ an oriented sub-bundle of codimension 1, with Frobenius form $\Lambda^2 B \xrightarrow{\Phi} TS/B$ non-degenerate. Then S is called **contact manifold**, $B \subset TS$ **the contact bundle**.

Definition 3: Let S be manifold of dimension $2k+1$, $B \subset TS$ an oriented sub-bundle of codimension 1. Assume that for any nowhere vanishing 1-form $\theta \in \Lambda^1 S$, the form $\theta \wedge (d\theta)^k$ is a non-degenerate volume form. Then (S, B) is called **a contact manifold**, and θ **a contact form**.

Proof. Step 1: (2) $\Leftrightarrow$ (3):
for each $x, y \in B$, $d\theta(x,y) = \theta([x,y]) = \Phi(x,y)$. Therefore, the Frobenius form $\Lambda^2 B \xrightarrow{\Phi} TS/B$ can be expressed as $\langle \Phi(x,y), \theta\rangle = d\theta(x,y)$. **Non-degeneracy of $\theta \wedge (d\theta)^k$ on TM is equivalent to non-degeneracy of $d\theta = \Phi$ on $B = \ker\theta$.** Therefore, $\Phi(x,y) = d\theta(x,y)$ is of maximal rank on B if and only if $\theta \wedge (d\theta)^k$ is non-degenerate.

Contact manifolds: three equivalent definitions (proofs, part two)

Definition 1: Let $C(S) = (S \times \mathbb{R}^{>}0)$ be a cone, equipped with the standard action $h_\lambda(x,t) = (x, \lambda t)$. Assume that $C(S)$ is equipped with a symplectic form ω such that $h_\lambda^* \omega = \lambda^2 \omega$. Then S is called **contact manifold**.

Step 2: (3) $\Rightarrow$ (1):
Let $M \xrightarrow{\pi} S$ be the space of positive vectors in the oriented 1-dimensional bundle $L := TS/B$, which is trivialized by the form θ, $V \in TM$ the unit vertical vector field, and $t: M \longrightarrow \mathbb{R}$ a map which associates $\theta(v)$ to a point $(s,v) \in M$, $s \in S, v \in L|_x$. Let $T := t\pi^*\theta \in \Lambda^1 M$, and let $\omega := dT$. Consider the vector field $r = tV \in TM$. Clearly, $\operatorname{Lie}_r T = 2T$, giving $\operatorname{Lie}_r dT = 2dT$. **To prove that M is a symplectic cone of S, it remains to show that dT is symplectic.**

Step 3: (3) $\Rightarrow$ (1), second part:
Since $\ker dt = \pi^* S$, any vector field $X \in TS$ can be naturally lifted to a vector field $\pi^{-1}(X) \in \ker dt \subset TM$. For each $Y := \pi^{-1}(y), x, y \in B$, one has $dT(X,Y) = T([X,Y]) = T(\pi^{-1}([x,y]))$, hence **$dT$ is non-degenerate on $\pi^{-1}(B)$.** Also, $dT \lrcorner V = T$, and $\ker T = \langle \pi^{-1} B, V \rangle$, hence **$dT$ is non-degenerate on the symplectic orthogonal complement to $\pi^{-1} B$.**

Contact manifolds: three equivalent definitions (proofs, part three)

Definition 1: Let $C(S) = (S \times \mathbb{R}^{>}0)$ be a cone, equipped with the standard action $h_\lambda(x,t) = (x, \lambda t)$. Assume that $C(S)$ is equipped with a symplectic form ω such that $h_\lambda^*\omega = \lambda^2\omega$. Then S is called **contact manifold**, and $C(S)$ **the symplectic cone.**

Definition 3: Let S be manifold of dimension $2k+1$, $B \subset TS$ an oriented sub-bundle of codimension 1. Assume that for any nowhere vanishing 1-form $\theta \in \Lambda^1 S$, the form $\theta \wedge (d\theta)^k$ is a non-degenerate volume form. Then (S, B) is called **a contact manifold**, and θ **a contact form**.

Step 4: (1) $\Rightarrow$ (3):
Let $M = C(S) = S \times \mathbb{R}^{>0}$, and $t \in C^\infty M$ the standard coordinate along $\mathbb{R}^{>0}$. Consider the vector field $r := t\frac{d}{dt}$, and the form $\theta := \omega \lrcorner r$. Since $\theta \lrcorner r = 0$ and

$$\mathrm{Lie}_r\, t^{-1}\theta = d(t^{-1}\theta)\lrcorner r + d(\theta \lrcorner r) = t^{-1}\theta - t^{-1}\theta + d(1) = 0,$$

the form $t^{-1}\theta$ is basic with respect to the projection $C(S) \longrightarrow S$. This gives a form θ on S. Finally, $(d\theta)^{k+1}$ is non-degenerate because $d\theta$ is symplectic. **Therefore, $(d\theta)^{k+1}\lrcorner r = (k+1)(d\theta)^k \wedge \theta$ is non-degenerate on S.**

■

Kähler manifolds.

Definition: Let (M, I) be a complex manifold, $\dim_{\mathbb{C}} M = n$, and g is Riemannian form. Then g is called **Hermitian** if $g(Ix, Iy) = g(x, y)$.

Remark: Since $I^2 = -\operatorname{Id}$, it is equivalent to $g(Ix, y) = -g(x, Iy)$. The form $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

Definition: The differential form ω is called **the Hermitian form of** (M, I, g).

Definition: A complex Hermitian manifold is called **Kähler** if $d\omega = 0$.

Remark: **Kähler manifolds are the main object of complex algebraic geometry** (algebraic geometry over $\mathbb{C}$). See e.g. Griffiths, Harris, *"Principles of Algebraic Geometry"*.

Examples of Kähler manifolds.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$.** The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$.

Remark: For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_x\mathbb{C}P^n$ is $U(n)$-invariant, hence unique up to a constant.

Claim: Fubini-Study form is Kähler. Indeed, $d\omega|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish (invariants of $U(n)$ are known since XIX century).

Corollary: Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler.

Almost complex manifolds.

A differential-geometric way of looking at Kähler manifolds.

Definition: **An almost complex structure** on a manifold M is an operator $I: TM \longrightarrow TM$ such that $I^2 = -\operatorname{Id}$. It is called **integrable** if I is induced by a complex structure.

Theorem: A Riemannian almost complex Hermitian manifold (M, I, g) is **Kähler if and only if $\nabla\omega = 0$,** where ∇ is a Levi-Civita connection.

Remark: This theorem is difficult (both ways). Integrability of almost complex structures takes some intensive work on PDEs. The implication $d\omega = 0$ $\Rightarrow \nabla\omega = 0$ is also non-trivial, but essentially linear-algebraic.

Remark: **One may think of Kähler manifolds as of symplectic manifolds with a Riemannian structure compatible with a symplectic form.** Locally, every symplectic manifold admits a Kähler structure (Darboux).

Sasakian manifolds.

Definition: Let M be a smooth manifold, $\dim M = 2n-1$, and (ω, I) a Kaehler structure on $M \times \mathbb{R}^{>0}$. Suppose that ω is **homogeneous**: $\Psi_q^*\omega = q^2 g$, where $\Psi_q(m,t) = (m, qt)$, and I is Ψ_q-invariant. Then M is called **Sasakian**, and $M \times \mathbb{R}^{>0}$ its **Kähler cone**.

Sasakian geometry is an odd-dimensional counterpart to Kähler geometry

Remark: A Sasakian manifold is obviosly contact. Indeed, **a Sasakian manifold is a contact manifold equipped with a compatible Riemannian metric.**

Example: An odd-dimensional sphere S^{2n-1} is Sasakian. Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{C}^n \backslash 0$ has the standard Kähler form $\sqrt{-1}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$ which is obviously homogeneous.

S. Sasaki, "On differentiable manifolds with certain structures which are closely related to almost contact structure", Tohoku Math. J. 2 (1960), 459-476.

Shigeo Sasaki (1912-1987).

Kenmotsu Katsuei, Sato Hajime, Sasaki Shigeo, 1980

Reeb field

DEFINITION: Let S be a Sasakian manifold, ω the Kähler form on $C(S)$, and $r = t\frac{d}{dt}$ the homothety vector field. Then $\operatorname{Lie}_{Ir} t = \langle dt, Ir\rangle = 0$, hence $I(r)$ is tangent to $S \subset C(S)$. This vector field is called **the Reeb field** of a Sasakian manifold.

REMARK: The Reeb field is dual to the contact form $\theta = \omega \lrcorner r$.

THEOREM: The Reeb field acts on a Sasakian manifold by contact isometries.

(see the next slide)

DEFINITION: A Sasakian manifold is called **regular** if the Reeb field generates a free action of S^1, **quasiregular** if all orbits of Reeb are closed, and **irregular** otherwise.

Reeb field acts by contact isometries

THEOREM: The Reeb field acts on a Sasakian manifold by contact isometries.

Proof. Step 1: Let $(C(S), \omega)$ be the cone of a Sasakian manifold with its Kähler form, and t the standard coordinate function. A **holomorphic vector field** is a vector field v such that its diffeomorphism flow e^{tv} is holomorphic. The homothety vector field $r = d\frac{d}{dt}$ is holomorphic, because $\operatorname{Lie}_r \tilde{\omega} = 2\tilde{\omega}$, $\operatorname{Lie}_r g = 2g$, giving $\operatorname{Lie}_r I = \operatorname{Lie}_r g\omega^{-1} = 0$.

Step 2: If Y is a holomorphic vector field, then IY is also holomorphic. To see this, chose (locally) a Kähler metric. Then $\nabla_{IZ}(Y) = I(\nabla_Y(Z)) + [Y, IZ]$ and $\nabla_Y(Z) + [Y, Z] = \nabla_Z Y$, showing that $\nabla_{IZ}(Y) = I(\nabla_Z(Y)) \Leftrightarrow Y$ is holomorphic. Then **the Reeb field acts on $C(X)$ holomorphically**.

Step 3: $\operatorname{Lie}_{\mathsf{Reeb}} \omega = d(\tilde{\omega} \lrcorner Ir) = d(tdt) = 0$. Therefore, $\operatorname{Lie}_{\mathsf{Reeb}} \omega = 0$. Since $\operatorname{Lie}_{\mathsf{Reeb}} I = 0$ as well, this implies that Reeb **is Killing.**

Step 4: Contact sub-bundle $B \subset TS$ is defined as $\ker \omega \lrcorner \frac{d}{dt}$; since **the Reeb field preserves t and ω, it preserves the contact sub-bundle.** ■

Quasiregular Sasakian manifolds.

Definition: Given a contact manifold (M, θ) with a Riemannian structure g, the dual vector field $\theta^\sharp$ is called **the Reeb field** of (M, θ, g).

Remark: For any Sasakian manifold, **the Reeb field generates a flow of diffeomorphisms acting on M by contact isometries**. This is obvious from the definition, because the Reeb field $\theta^\sharp = It\frac{d}{dt}$ acts by holomorphic isometries on the Kähler cone.

Definition: A Sasakian manifold M is called **quasiregular** if all orbits of the Reeb flow are compact. The space of orbits of the Reeb flow is a complex orbifold. **Every quasiregular Sasakian manifold is a total space of S^1-bundle over a complex orbifold.**

This is easy to see, because the quotient of M over the Reeb flow is the same as the quotient of CM over its complexification, generated by $\theta^\sharp$ and $I\theta^\sharp$.

Examples of Sasakian manifolds.

Example: Let $X \subset \mathbb{C}P^n$ be a complex submanifold, and $CX \subset \mathbb{C}^{n+1}\backslash 0$ the corresponding cone. The cone CX is obviously Kähler and homogeneous, hence **the intersection $CX \cap S^{2n-1}$ is Sasakian.** This intersection is an S^1-bundle over X. This construction gives many interesting contact manifolds, including Milnor's exotic 7-spheres, which happen to be Sasakian.

Remark: A link of a homogeneous singularity is always Sasakian.

Remark: Every quasiregular Sasakian manifold is obtained this way, for some Kähler metric on $\mathbb{C}^{n+1}$.

Remark: Every Sasakian manifold can be deformed to a quasiregular one.

CR-manifolds.

Definition: Let M be a smooth manifold, $B \subset TM$ a sub-bundle in a tangent bundle, and $I: B \longrightarrow B$ an endomorphism satisfying $I^2 = -1$. Consider its $\sqrt{-1}$ -eigenspace $B^{1,0}(M) \subset B \otimes \mathbb{C} \subset T_C M = TM \otimes \mathbb{C}$. Suppose that $[B^{1,0}, B^{1,0}] \subset B^{1,0}$. Then (B, I) is called **a CR-structure on M**.

Example: A complex manifold is CR, with $B = TM$. Indeed, $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$ **is equivalent to integrability of the complex structure** (Newlander-Nirenberg).

Example: Let X be a complex manifold, and $M \subset X$ a hypersurface. Then $B := \dim_{\mathbb{C}} TM \cap I(TM) = \dim_{\mathbb{C}} X - 1$, hence $\operatorname{rk} B = n - 1$. Since $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$, **$M$ is a CR-manifold.**

Definition: **A Frobenius form of a CR-manifold** is the tensor $B \otimes B \longrightarrow TM/B$ mapping X, Y to the $\Pi_{TM/B}([X, Y])$. It is an obstruction to integrability of the foliation given by B.

Contact CR-manifolds.

Complex algebraic geometry is a rich source of contact structures.

Definition: Let (M, B, I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a contact CR-manifold** if its Frobenius form is non-degenerate.

Remark: Since $[B^{1,0}, B^{1,0}] \subset B^{1,0}$ and $[B^{0,1}, B^{0,1}] \subset B^{0,1}$, the Frobenius form is a pairing between $B^{0,1}$ and $B^{1,0}$. This means that it is Hermitian.

Definition: Let (M, B, I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a strictly pseudoconvex CR-manifold** if its Frobenius form is positive definite everywhere.

Example: Let h be a function on a complex manifold such that $\partial\bar\partial h = \omega$ is a positive definite Hermitian form, and $X = h^{-1}(c)$ its level set. Then the Frobenius form of X is equal to $\omega|_X$. In particular, X **is a strictly pseudoconvex CR-manifold**.

CR-geometry of Sasakian manifolds.

Claim: Let M be a Sasakian manifold, $CM = M \times \mathbb{R}^{>0}$ its Kähler cone, and $\varphi(m,t) = t$ the projection of CM to $\mathbb{R}^{>0}$. Then $\sqrt{-1}\,\partial\bar\partial\varphi = \omega$ **is its Kähler form.**

Proof:
$\sqrt{-1}\,\partial\bar\partial\varphi = \frac{1}{2}dd^c\varphi = \frac{1}{2}dId\varphi = dId(t^2) = d(\omega \lrcorner t\frac{d}{dt}) = \omega$ as we have already seen.
■

Corollary:
A Sasakian manifold is strictly pseudoconvex as a CR-manifold.

Question:
Which strictly pseudoconvex CR-manifolds admit Sasakian structures?

Answer: Let M be a compact, strictly pseudoconvex CR-manifold. Then M admits a Sasakian structure **if and only if M admits a CR-holomorphic vector field, which is everywhere transversal to B.** Moreover, this vector field becomes the Reeb field for this Sasakian structure and the Sasakian structure on M **is uniquely determined by the CR-structure and the Reeb field.**