Sasakian manifolds and CR-geometry

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Estruturas geométricas em variedades

February 16, 2023.
IMPA

Contact manifolds.

Definition: Let M be a smooth manifold, $\dim M = 2n-1$, and ω a symplectic form on $M \times \mathbb{R}^{>0}$. Suppose that ω is **automorphic**: $\Psi_q^* \omega = q^2 \omega$, where $\Psi_q(m,t) = (m, qt)$. Then M is called **contact**.

DEFINITION: **The contact form** on M is defined as $\theta = \omega \lrcorner \vec{T}$, where $\vec{T} = t\frac{d}{dt}$. Then $d\theta = [d, \cdot \lrcorner \vec{T}]\omega = \operatorname{Lie}_{\vec{T}} \omega = \omega$. Therefore, **the form $(d\theta)^{n-1} \wedge \theta = \frac{1}{n} \omega^n \lrcorner \vec{T}$ is non-degenerate on $M \times \{t_0\} \subset M \times \mathbb{R}^{>0}$.**

Remark: Usually, a contact manifold is defined as a **$(2n-1)$-manifold with 1-form θ such that $d\theta^{n-1} \wedge \theta$ is nowhere degenerate.**

Example: **An odd-dimensional sphere S^{2n-1} is contact.** Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{R}^{2n} \backslash 0$ has the standard symplectic form $\sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}$ which is obviously homogeneous.

Contact geometry is an odd-dimensional counterpart to symplectic geometry

DEFINITION: **Reeb field** on a contact manifold (M, θ) is a field $R \in TM$ such that $d\theta(R, \cdot) = 0$ and $\langle \theta, R \rangle = 1$.

Boothby-Wang theorem

W. M. Boothby, H. C. Wang, On Contact Manifolds Annals of Mathematics, Second Series, Vol. 68, No. 3 (Nov., 1958), pp. 721-734.

DEFINITION: A contact manifold (M, θ) is **normal** if it is equipped with an S^1-action preserving θ and tangent to the Reeb field.

REMARK: Let (M, θ) be a contact manifold. **Then the form $d\theta$ is non-degenerate the bundle $\ker \theta \subset TM$.**

THEOREM: (Boothby-Wang, 1958)
Let (M, θ) be a normal contact manifold. **Then its space X of Reeb orbits is symplectic and the natural projection $\pi : M \longrightarrow X$ induces a symplectic isomorphism $d\pi : \ker \theta|_x \longrightarrow T_{\pi(x)}X$.**

THEOREM: (Boothby-Wang, 1958)
Let (M, θ) be a normal contact manifold, and (X, ω) the symplectic manifold obtained as its space of Reeb orbits. **Then the cohomology class of ω is integral.** Conversely, for any symplectic manifold (X, ω) with $[\omega] \in H^2(X, \mathbb{Z})$, **there exists a principal S^1-bundle L with $c_1(L) = [\omega]$ and a normal contact structure on $M := \operatorname{Tot}(L)$ such that the corresponding Boothby-Wang projection coincides with the natural map $M \longrightarrow X$.**

Kähler manifolds.

Definition: Let (M, I) be a complex manifold, $\dim_{\mathbb{C}} M = n$, and g is Riemannian form. Then g is called **Hermitian** if $g(Ix, Iy) = g(x, y)$.

Remark: Since $I^2 = -\operatorname{Id}$, it is equivalent to $g(Ix, y) = -g(x, Iy)$. **The form $\omega(x, y) := g(x, Iy)$ is skew-symmetric.**

Definition: The differential form ω is called **the Hermitian form of (M, I, g)**.

Definition: A complex Hermitian manifold is called **Kähler** if $d\omega = 0$.

Sasakian manifolds.

Definition: Let (M, g_M) be a Riemannian manifold, $\dim M = 2n-1$, and (g, ω, I) a Kaehler structure on $M \times \mathbb{R}^{>0}$ with $g = g_M + t^2 dt \otimes dt$. Suppose that ω is **automorphic**: $\Psi_q^* \omega = q^2 g$, where $\Psi_q(m,t) = (m, qt)$, and I is Ψ_q-invariant. Then M is called **Sasakian**, and $M \times \mathbb{R}^{>0}$ its **Kähler cone**.

Sasakian geometry is an odd-dimensional counterpart to Kähler geometry

Remark: A Sasakian manifold is obviosly contact. Indeed, **a Sasakian manifold is a contact manifold equipped with a compatible Riemannian metric.**

Example: An odd-dimensional sphere S^{2n-1} is Sasakian. Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{C}^n \backslash 0$ has the standard Kähler form $\sqrt{-1} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$ which is obviously automorphic.

S. Sasaki, "On differentiable manifolds with certain structures which are closely related to almost contact structure", Tohoku Math. J. 2 (1960), 459-476.

Kähler potentials on the Kähler cones

DEFINITION: A **Kähler potential** on a Kähler manifold (M, I, ω) is a function f such that the (1,1)-form $dd^c(f)$ is equal to ω, where $dd^c = dIdI^{-1} = \frac{\partial\bar\partial}{-2\sqrt{-1}}$.

CLAIM: Let M be a Sasakian manifold and $C(M) = M \times \mathbb{R}^{>0}$ its Kähler cone, with t the parameter on $\mathbb{R}^{>0}$ and ω its Kähler form. **Then $dd^c(t^2) = \omega$, in other words, $\frac{1}{2}t^2$ is a Kähler potential.**

Proof: Let $\vec{r} = td/dt$ be the radial vector field. Cartan's formula $\operatorname{Lie}_X \eta = (d\eta) \lrcorner X + d(\eta \lrcorner X)$ gives

$$2\omega = \operatorname{Lie}_{\vec{r}}\omega = d(\omega \lrcorner \vec{r}) = d(tI(dt)) = 2dd^c(t^2).$$

■

COROLLARY: The field $I(\vec{r})$ **acts on $C(M)$ by holomorphic isometries.**

Proof: Indeed, $\vec{r}$ is holomorphic, and $\operatorname{Lie}_{I\vec{r}}(t) = \langle dt, Itd/dt \rangle = 0$. Therefore, $I(\vec{t})$ preserves the Kähler potential and the Kähler structure. ■

Reeb field

Definition: Given a contact manifold (M,θ), a vector field R is called **the Reeb field** of (M,θ), if $d\theta \lrcorner R = 0$ and $\theta(R) = 1$.

DEFINITION: Let $C(M) := M \times \mathbb{R}^{>0}$ be the cone of a Sasakian manifold M. The vector field $\vec{r} := td/dt$ is called **the Lee field** on $C(M)$. Clearly, $\vec{r}$ acts on $C(M)$ by holomorphic homotheties.

REMARK: On a Sasakian manifold, $d\theta$ is restriction of the Kähler form to $M = M \times \{1\} \subset M \times \mathbb{R}^{>0}$, hence $\ker d\theta|_M = I(R^c)$. Also, $\theta(I(\vec{r}))|_M = 1$. **Therefore, $R := I(\vec{r})$ is the Reeb field of M.**

COROLLARY: The Reeb field of a Sasakian manifold **is the Riemannian dual to its contact form $\theta = \omega(\vec{r}, \cdot)$.**

COROLLARY: For any Sasakian manifold, **the Reeb field generates a flow of diffeomorphisms acting on M by contact isometries**.

Proof: The Reeb field $\theta^\sharp = I(\vec{r})$ acts holomorphically because $\vec{r}$ acts holomorphically, and preserves the Kähler potentialm as shown above. ■

Quasiregular Sasakian manifolds

Definition: A Sasakian manifold M is called **quasiregular** if all orbits of the Reeb flow are compact.

The following claim is **the Sasakian version of Boothby-Wang theorem.**

CLAIM: Every quasiregular Sasakian manifold is a total space of S^1-bundle over a Kähler orbifold.

Proof: The quotient of M over the Reeb flow is the same as the quotient of its cone $C(M)$ over its complexification, generated by $\vec{r}$ and $R = I(\vec{r})$. ■

REMARK: The space of Reeb orbits X of a quasiregular Sasakian manifold **is in fact projective,** as follows from Kodaira theorem. Indeed, $C(M)$ is $\mathbb{C}^*$-bundle over X, and the corresponding line bundle has positive curvature which can be expressed as $dd^c \log(\varphi)$, where $\varphi = r^2$ is the Kähler potential of $C(M)$.

Examples of Sasakian manifolds.

Example: Let $X \subset \mathbb{C}P^n$ be a complex submanifold, and $CX \subset \mathbb{C}^{n+1}\backslash 0$ the corresponding cone. The cone CX is obviously Kähler and automorphic, hence **the intersection $CX \cap S^{2n-1}$ is Sasakian.** This intersection is an S^1-bundle over X. This construction gives many interesting contact manifolds, including Milnor's exotic 7-spheres, which happen to be Sasakian.

Remark: A link of a homogeneous singularity is always Sasakian.

Remark: Every quasiregular Sasakian manifold is obtained this way.

Remark: All 3-dimensional Sasakian manifolds, except S^3 and the quotients $\frac{S^3}{\mathbb{Z}/n}$, are quasiregular (H. Geiges, 1997, F. Belgun, 2000).

Remark: Every Sasakian manifold is diffeomorphic to a quasiregular one.

CR-manifolds.

Definition: Let M be a smooth manifold, $B\subset TM$ a sub-bundle in a tangent bundle, and $I: B\longrightarrow B$ an endomorphism satisfying $I^2=-1$. Consider its $\sqrt{-1}$-eigenspace $B^{1,0}(M)\subset B\otimes\mathbb{C}\subset T_C M=TM\otimes\mathbb{C}$. Suppose that $[B^{1,0},B^{1,0}]\subset B^{1,0}$. Then (B,I) is called **a CR-structure on M**.

Example: A complex manifold is CR, with $B=TM$. Indeed, $[T^{1,0}M,T^{1,0}M]\subset T^{1,0}M$ **is equivalent to integrability of the complex structure** (Newlander-Nirenberg).

Example: Let X be a complex manifold, and $M\subset X$ a hypersurface. Then $B:=\dim_{\mathbb{C}} TM\cap I(TM)=\dim_{\mathbb{C}} X-1$, hence $\operatorname{rk} B=n-1$. Since $[T^{1,0}X,T^{1,0}X]\subset T^{1,0}X$, **$M$ is a CR-manifold.**

Definition: **A Frobenius form of a CR-manifold** is the tensor $B\otimes B\longrightarrow TM/B$ mapping X,Y to the projection $\Pi_{TM/B}([X,Y])$. It is an obstruction to integrability of the foliation given by B.

Contact CR-manifolds.

Definition: Let (M, B, I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a contact CR-manifold** if its Frobenius form is non-degenerate.

Remark: Since $[B^{1,0}, B^{1,0}] \subset B^{1,0}$ and $[B^{0,1}, B^{0,1}] \subset B^{0,1}$, the Frobenius form is a pairing between $B^{0,1}$ and $B^{1,0}$. **This means that it is of Hodge type (1,1), that us, pseudo-Hermitian.**

Definition: Let (M, B, I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a strictly pseudoconvex CR-manifold** if its Frobenius form is positive definite everywhere.

Example: Let h be a function on a complex manifold such that $\partial\bar\partial h = \omega$ is a positive definite Hermitian form, and $X = h^{-1}(c)$ its level set. Then the Frobenius form of X is equal to $\omega|_X$. In particular, X **is a strictly pseudoconvex CR-manifold**.

CR-geometry of Sasakian manifolds.

THEOREM: A Sasakian manifold is strictly pseudoconvex as a CR-manifold.

Proof: Let $\varphi = t^2$ be the Kähler potential on $C(M) = M \times \mathbb{R}^{>0}$. Then M is its level set. **A level set of a Kähler potential is always strictly pseudoconvex.** ■

Question: Which strictly pseudoconvex CR-manifolds admit Sasakian structures?

Answer: (Ornea-V.) Let (M, B, I) be a compact, strictly pseudoconvex CR-manifold. **Then M admits a Sasakian metric if and only if M admits a CR-holomorphic vector field which is transversal to B.** Moreover, **for every such field v, there exists a unique Sasakian metric such that v or $-v$ is its Reeb field.**

CR-Holomorphic vector fields on pseudoconvex CR-manifolds

THEOREM: Let S be a strictly pseudoconvex compact CR-manifold, $\dim_{\mathbb{R}} S \geqslant 5$. **Then S admits a Sasakian structure if and only if S admits a transversal CR-holomorphic vector field.** This vector field becomes a Reeb field of this Sasakian manifold.

Step 1: By Rossi-Andreotti-Siu, $S = \partial M$, where $M = \operatorname{Spec}(H^0(\mathcal{O}_S)_b)$ is a Stein variety with isolated singularities, and R acts on M by holomorphic automorphisms.

Step 2: Since R is transversal to the contact distribution, IR is transversal to ∂M; replacing R by $-R$, we can always assume that IR points toward interior of M, and $A_\varepsilon := e^{\varepsilon IR}$ **for small ε maps M to a subset $A_\varepsilon(M) \subset M$ with compact closure.**

Step 3: Consider the ring $\mathcal{H} = H^0(\mathcal{O}_M)_b$ of bounded holomorphic functions on M, with sup-metric. Then $\mathcal{H}$ is a Banach ring. Since $A_\varepsilon(M)$ has compact closure, $A_\varepsilon^*(B)$ **is a normal family, where B is any open ball in the Banach space $\mathcal{H}$, and A_ε^* is a compact operator.**

CR-Holomorphic vector fields on pseudoconvex CR-manifolds (2)

Step 4: By maximum principle, for any non-constant $f \in \mathcal{H}$, one has $\sup_{A_\varepsilon(M)} |f| < \sup_M |f|$. **Since any limit point $f_{\lim}$ of a sequence $(A_\varepsilon^i)^* f$ satisfies $\sup A_\varepsilon(M)|f| = \sup_M |f|$, it is constant.** A limit function $f_{\lim}$ exists, because $A_\varepsilon^*(B)$ is precompact.

Step 5: This implies that for each $z \in M$, a limit point $z_{\lim}$ of a sequence $\{z, A_\varepsilon z, A_\varepsilon^2 z, ...\}$ is unique and independent of z. Indeed, $f_{\lim}(z) = f(z_{\lim})$, but $f_{\lim} = const$. This implies that **A_ε is a holomorphic contraction contracting M to the origin point $x_0 \subset M$.**

Step 6: Since $R\Big|_{S_\varepsilon} = A_\varepsilon(R)$ is nowhere vanishing for each ε, the vector field $\vec{r} := IR$ is transversal to $S_\varepsilon := A_\varepsilon(S)$ pointing to the origin. Therefore, through each point of S passes a unique solution $\rho(t)$ of an equation $\frac{d\rho(t)}{dt} = \vec{r}$.

Step 7: Let φ_λ be a ρ-automorphic Kähler potential which is equal to 1 on $S \subset M$. Such a potential exists, because S is strictly pseudoconvex, hence it can be realized as a level set of a psh function. **The Lie algebra $\langle R, IR\rangle$ acts on $(M, dd^c\varphi_\lambda)$ by holomorphic homotheties, hence it is a conical Kähler manifold** (Kamishima-Ornea). Therefore, S is Sasakian. ■

Quasi-regular deformations of Sasakian manifolds.

THEOREM: (D. Burns) Let (M, B, I) be a pseudoconvex CR-manifold, which is not equivalent to a sphere with the standard CR-structure. **Then the group of holomorphic automorphisms of M is compact.**

THEOREM: (Ornea-V.) Let (M, B, I) be a CR manifold admitting a Sasakian structure, that is, a CR-holomorphic vector field R which is transversal to B. **Then $R = \lim_i R_i$, where R_i are Reeb fields of quasiregular Sasakian structures.**

Proof. Step 1: Let G be the closure of the Lie subgroup in $\operatorname{Aut}(M, B, I)$ generated by e^{tR}. Since G is compact and commutative, it is a torus. For any vector field $R' \in \operatorname{Lie}(G)$ in its Lie algebra sufficiently close to R, it is also transversal to B, hence gives another Sasakian structure.

Step 2: A Reeb field $R' \in \operatorname{Lie}(G)$ is quasiregular if and only if it generates a compact subgroup, that is, is rational with respect to the rational structure on the Lie algebra $\operatorname{Lie} G$. However, rational points are dense in $\operatorname{Lie} G$. ■