Closed Reeb orbits in Sasakian manifolds

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Contact manifolds.

Definition: Let M be a smooth manifold, $\dim M = 2n-1$, and ω a symplectic form on $M \times \mathbb{R}^{>0}$. Suppose that ω is **automorphic**: $\Psi_q^*\omega = q^2\omega$, where $\Psi_q(m,t) = (m, qt)$. Then M is called **contact**.

Remark: The contact form on M is defined as $\theta = \omega \lrcorner \vec{T}$, where $\vec{T} = t\frac{d}{dt}$. Then $d\theta = [d, \cdot \lrcorner \vec{T}]\omega = \operatorname{Lie}_{\vec{T}}\omega = \omega$. Therefore, **the form $d\theta^{n-1} \wedge \theta = \frac{1}{n}\omega^n \lrcorner \vec{T}$ is non-degenerate on $M \times \{t_0\} \subset M \times \mathbb{R}^{>0}$.**

Remark: Usually, a contact manifold is defined as a **$(2n-1)$-manifold with 1-form θ such that $d\theta^{n-1} \wedge \theta$ is nowhere degenerate.**

Example: An odd-dimensional sphere S^{2n-1} is contact. Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{R}^{2n}\backslash 0$ has the standard symplectic form $\sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}$ which is obviously homogeneous.

Contact geometry is an odd-dimensional counterpart to symplectic geometry

Kähler manifolds.

Definition: Let (M, I) be a complex manifold, $\dim_{\mathbb{C}} M = n$, and g is Riemannian form. Then g is called **Hermitian** if $g(Ix, Iy) = g(x, y)$.

Remark: Since $I^2 = -\operatorname{Id}$, it is equivalent to $g(Ix, y) = -g(x, Iy)$. **The form $\omega(x, y) := g(x, Iy)$ is skew-symmetric.**

Definition: The differential form ω is called **the Hermitian form of (M, I, g)**.

Definition: A complex Hermitian manifold is called **Kähler** if $d\omega = 0$.

Sasakian manifolds.

Definition: Let (M, g_M) be a Riemannian manifold, $\dim M = 2n-1$, and (g, ω, I) a Kaehler structure on $M \times \mathbb{R}^{>0}$ with $g = g_M + t^2 dt \otimes dt$. Suppose that ω is **automorphic**: $\Psi_q^* \omega = q^2 g$, where $\Psi_q(m,t) = (m, qt)$, and I is Ψ_q-invariant. Then M is called **Sasakian**, and $M \times \mathbb{R}^{>0}$ its **Kähler cone**.

Sasakian geometry is an odd-dimensional counterpart to Kähler geometry

Remark: A Sasakian manifold is obviosly contact. Indeed, **a Sasakian manifold is a contact manifold equipped with a compatible Riemannian metric.**

Example: An odd-dimensional sphere S^{2n-1} is Sasakian. Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{C}^n \backslash 0$ has the standard Kähler form $\sqrt{-1} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$ which is obviously automorphic.

S. Sasaki, "On differentiable manifolds with certain structures which are closely related to almost contact structure", Tohoku Math. J. 2 (1960), 459-476.

Kähler potentials on the Kähler cones

DEFINITION: A **Kähler potential** on a Kähler manifold (M, I, ω) is a function f such that the (1,1)-form $dd^c(f)$ is equal to ω, where $dd^c = dIdI^{-1} = \frac{\partial\bar\partial}{-2\sqrt{-1}}$.

CLAIM: Let M be a Sasakian manifold and $C(M) = M \times \mathbb{R}^{>0}$ its Kähler cone, with t the parameter on $\mathbb{R}^{>0}$ and ω its Kähler form. **Then $dd^c(t^2) = \omega$, in other words, $\frac{1}{2}t^2$ is a Kähler potential.**

Proof: Let $\vec{r} = td/dt$ be the radial vector field. Cartan's formula $\operatorname{Lie}_X \eta = (d\eta) \lrcorner X + d(\eta \lrcorner X)$ gives

$$2\omega = \operatorname{Lie}_{\vec{r}}\omega = d(\omega \lrcorner \vec{r}) = d(tI(dt)) = 2dd^c(t^2).$$

■

COROLLARY: The field $I(\vec{r})$ **acts on $C(M)$ by holomorphic isometries.**

Proof: Indeed, $\vec{r}$ is holomorphic, and $\operatorname{Lie}_{I\vec{r}}(t) = \langle dt, Itd/dt \rangle = 0$. Therefore, $I(\vec{t})$ preserves the Kähler potential and the Kähler structure. ■

Reeb field

Definition: Given a contact manifold (M, θ), a vector field R is called **the Reeb field** of (M, θ), if $d\theta \lrcorner R = 0$ and $\theta(R) = 1$.

DEFINITION: Let $C(M) := M \times \mathbb{R}^{>0}$ be the cone of a Sasakian manifold M. The vector field $\vec{r} := td/dt$ is called **the Lee field** on $C(M)$. Clearly, $\vec{r}$ acts on $C(M)$ by holomorphic homotheties.

REMARK: On a Sasakian manifold, $d\theta$ is restriction of the Kähler form to $M = M \times \{1\} \subset M \times \mathbb{R}^{>0}$, hence $\ker d\theta|_M = I(R^c)$. Also, $\theta(I(\vec{r}))|_M = 1$. **Therefore, $R := I(\vec{r})$ is the Reeb field of M.**

COROLLARY: The Reeb field of a Sasakian manifold **is the Riemannian dual to its contact form $\theta = \omega(\vec{r}, \cdot)$.**

COROLLARY: For any Sasakian manifold, **the Reeb field generates a flow of diffeomorphisms acting on M by contact isometries**.

Proof: The Reeb field $\theta^\sharp = I(\vec{r})$ acts holomorphically because $\vec{r}$ acts holomorphically, and preserves the Kähler potentialm as shown above. ■

Quasiregular Sasakian manifolds

Definition: A Sasakian manifold M is called **quasiregular** if all orbits of the Reeb flow are compact.

CLAIM: Every quasiregular Sasakian manifold is a total space of S^1-bundle over a complex orbifold.

Proof: The quotient of M over the Reeb flow is the same as the quotient of its cone $C(M)$ over its complexification, generated by $\vec{r}$ and $R = I(\vec{r})$. ■

REMARK: The space of Reeb orbits X of a quasiregular Sasakian manifold **is in fact projective,** as follows from Kodaira theorem. Indeed, $C(M)$ is $\mathbb{C}^*$-bundle over X, and the corresponding line bundle has positive curvature which can be expressed as $dd^c \log(\varphi)$, where $\varphi = r^2$ is the Kähler potential of $C(M)$.

Examples of Sasakian manifolds.

Example: Let $X \subset \mathbb{C}P^n$ be a complex submanifold, and $CX \subset \mathbb{C}^{n+1}\backslash 0$ the corresponding cone. The cone CX is obviously Kähler and automorphic, hence **the intersection $CX \cap S^{2n-1}$ is Sasakian.** This intersection is an S^1-bundle over X. This construction gives many interesting contact manifolds, including Milnor's exotic 7-spheres, which happen to be Sasakian.

Remark: A link of a homogeneous singularity is always Sasakian.

Remark: Every quasiregular Sasakian manifold is obtained this way.

Remark: All 3-dimensional Sasakian manifolds are quasiregular, except S^3 (H. Geiges, 1997, F. Belgun, 2000).

Remark: **Every Sasakian manifold is diffeomorphic to a quasiregular one** (Ornea-V., arXiv:math/0306077)

Number of closed Reeb orbits

Every Sasakian manifold can be approximated by a sequence of quasiregular ones.

THEOREM: (Ornea-V., arXiv:math/0306077) Let $\mathfrak{S}$ be a Sasakian structure on a manifold M. **Then $\mathfrak{S} = \lim_i \mathfrak{S}_i$, where $\mathfrak{S}_i$ are quasiregular Sasakian structures.**

The main result of today's talk (joint with Liviu Ornea).

THEOREM: Let M be a Sasakian manifold, M' a quasiregular Sasakian manifold approximating M as above, and X its space of Reeb orbits on M'. Let $N := \sum_i b_i(X)$ be the sum of its Betti numbers. **Then M has at least N closed Reeb orbits.**

REMARK: Since X is projective, $\sum_i b_i(X) \geqslant m+1$, where $m = \dim_{\mathbb{C}} X$ by Lefschetz theorem. Therefore, **the number of closed Reeb orbits on a Sasakian manifold M with $\dim_{\mathbb{R}} M = 2n+1$ is at least $n+1$.**

CR-manifolds.

Definition: Let M be a smooth manifold, $B \subset TM$ a sub-bundle in a tangent bundle, and $I: B \longrightarrow B$ an endomorphism satisfying $I^2 = -1$. Consider its $\sqrt{-1}$ -eigenspace $B^{1,0}(M) \subset B \otimes \mathbb{C} \subset T_C M = TM \otimes \mathbb{C}$. Suppose that $[B^{1,0}, B^{1,0}] \subset B^{1,0}$. Then (B, I) is called **a CR-structure on M**.

Example: A complex manifold is CR, with $B = TM$. Indeed, $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$ **is equivalent to integrability of the complex structure** (Newlander-Nirenberg).

Example: Let X be a complex manifold, and $M \subset X$ a hypersurface. Then $B := \dim_{\mathbb{C}} TM \cap I(TM) = \dim_{\mathbb{C}} X - 1$, hence $\operatorname{rk} B = n - 1$. Since $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$, **$M$ is a CR-manifold.**

Definition: **A Frobenius form of a CR-manifold** is the tensor $B \otimes B \longrightarrow TM/B$ mapping X, Y to the projection $\Pi_{TM/B}([X, Y])$. It is an obstruction to integrability of the foliation given by B.

Contact CR-manifolds.

Definition: Let (M, B, I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a contact CR-manifold** if its Frobenius form is non-degenerate.

Remark: Since $[B^{1,0}, B^{1,0}] \subset B^{1,0}$ and $[B^{0,1}, B^{0,1}] \subset B^{0,1}$, the Frobenius form is a pairing between $B^{0,1}$ and $B^{1,0}$. **This means that it is of Hodge type (1,1), that us, pseudo-Hermitian.**

Definition: Let (M, B, I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a strictly pseudoconvex CR-manifold** if its Frobenius form is positive definite everywhere.

Example: Let h be a function on a complex manifold such that $\partial\bar\partial h = \omega$ is a positive definite Hermitian form, and $X = h^{-1}(c)$ its level set. Then the Frobenius form of X is equal to $\omega|_X$. In particular, X **is a strictly pseudoconvex CR-manifold**.

CR-geometry of Sasakian manifolds.

THEOREM: A Sasakian manifold is strictly pseudoconvex as a CR-manifold.

Proof: Let $\varphi = t^2$ be the Kähler potential on $C(M) = M \times \mathbb{R}^{>0}$. Then M is its level set. **A level set of a Kähler potential is always strictly pseudoconvex.** ■

Question: Which strictly pseudoconvex CR-manifolds admit Sasakian structures?

Answer: (Ornea, V., arXiv:math/0606136) Let (M, B, I) be a compact, strictly pseudoconvex CR-manifold. **Then M admits a Sasakian metric if and only if M admits a CR-holomorphic vector field which is transversal to B.** Moreover, **for every such field v, there exists a unique Sasakian metric such that v or $-v$ is its Reeb field.**

Quasi-regular deformations of Sasakian manifolds.

THEOREM: (D. Burns) Let (M, B, I) be a pseudoconvex CR-manifold, which is not equivalent to a sphere with the standard CR-structure. **Then the group of holomorphic automorphisms of M is compact.**

THEOREM: (Ornea-V., arXiv:math/0306077) Let (M, B, I) be a CR manifold admitting a Sasakian structure, that is, a CR-holomorphic vector field R which is transversal to B. **Then $R = \lim_i R_i$, where R_i are Reeb fields of quasiregular Sasakian structures.**

Proof. Step 1: Let G be the closure of the Lie subgroup in $\mathrm{Aut}(M, B, I)$ generated by e^{tR}. Since G is compact and commutative, it is a torus. For any vector field $R' \in \mathrm{Lie}(G)$ in its Lie algebra sufficiently close to R, it is also transversal to B, hence gives another Sasakian structure.

Step 2: A Reeb field $R' \in \mathrm{Lie}(G)$ is quasiregular if and only if it generates a compact subgroup, that is, is rational with respect to the rational structure on the Lie algebra $\mathrm{Lie}\, G$. However, rational points are dense in $\mathrm{Lie}\, G$. ■

Closed Reeb orbits

CLAIM: Let M be a Sasakian manifold, and $G = \overline{\langle e^{tR} \rangle}$ the compact group constructed above. **Then each 1-dimensional orbit of G is a closed Reeb orbit**.

REMARK: Let $R_1 \in \operatorname{Lie}(G)$ be a rational Reeb field approximating R. The corresponding Sasakian structure is quasiregular, **and G acts on a projective orbifold $X = M/\langle e^{tR_1} \rangle$ by complex isometries.** Clearly, **each fixed point of G acting on X corresponds to 1-dimensional orbit of G acting on M.**

COROLLARY: In these assumptions, the number of closed Reeb orbit in M **is no smaller than the number of fixed points of G acting on X.**

Clearly, the number of fixed points of G acting on X is equal to the number of zeros of a general vector field $r \in \operatorname{Lie} G$. Now the main result is given by the following version of Bialynicki-Birula theorem.

THEOREM: Let r be a holomorphic vector field with isolated zeros on a compact projective orbifold X. **Then the number of zeros of r is equal to $\sum_i b_i(X)$.**

Proof: *C. Fontanari, Towards the cohomology of moduli spaces of higher genus stable maps. Pubblicato su Archiv der Mathematik 89 (2007), 530-535 (Proposition 1).* ■

Vaisman manifolds

DEFINITION: Let $C(M)$ be the cone of a compact Sasakian manifold, and q a non-isometric holomorphic homothety of $C(M)$. The quotient $V := C(M)/\langle q\rangle$, considered as a compact complex manifold, is called **a Vaisman manifold**.

REMARK: Consider a form $dd^c \log t$ on $C(M) = M \times \mathbb{R}^{>0}$, where t is the parameter on $\mathbb{R}^{>0}$. Then ω_0 is q-invariant. Moreover, $\omega_0 = \frac{1}{t^2}(\omega - dt \wedge I(dt))$ is semi-positive definite.

REMARK: The form $d\log t$ is already q-invariant. Therefore, the form ω_0 is well defined on each Vaisman manifold, and is exact there,

DEFINITION: A foliation $\Sigma := \ker \omega_0$ is called **canonical foliation** on a Vaisman manifold.

CLAIM: The canonical foliation **is independent from the choice of $C(M)$ and q.**

Proof: Suppose we have two different exact and (semi-)positive forms ω_0 and ω_0'. The sum $\omega_1\omega_0+\omega_0'$ is also exact and semipositive. Unless $\ker\omega_0 = \ker\omega_0$, this sum is strictly positive, which is impossible because $\int_V \omega_1^{\dim_{\mathbb{C}} V} = 0$ by Stoke's theorem (because ω_0 is exact). ■

Complex curves on Vaisman manifolds

THEOREM: Let C be a complex curve on a Vaisman manifold V. **Then C is a leaf of the canonical foliation**. In particular, **C is an elliptic curve**.

Proof: $\int_C \omega_0 = 0$ by Stoke's theorem (because ω_0 is exact), hence C is tangent to $\Sigma = \ker \omega_0$. However, all compact leaves of Σ are elliptic curves because its tangent bundle $T\Sigma$ is trivial. ■

REMARK: Let q act on the cone $C(M)$ of Sasakian manifold by $(m,t) \longrightarrow (m, \lambda t)$ with $\lambda \in \mathbb{R}^{>1}$. Then the canonical foliation is generated by $\vec{r}, I\vec{r}$. In particular, **a leaf of Σ is compact if and only if the corresponding orbit of $R = I\vec{r}$ is compact.**

We proved the following

COROLLARY: Let M be a Sasakian manifold, and $V := C(M)/\langle q \rangle$ the corresponding Vaisman manifold, with $q = \lambda \in \mathbb{R}^{>1}$. **Then the number of closed Reeb orbit is equal to the number of elliptic curves in V.**

Vaisman manifolds as submanifolds in Hopf manifolds

THEOREM: (Ornea-V., 2006) A compact complex manifold admits Vaisman metric if and only if V **admits a holomorphic embedding into a diagonal Hopf manifold** $H = (\mathbb{C}^n \backslash 0)/\langle A\rangle$, where A is a diagonal endomorphism with eigenvalues $(\alpha_1, ...\alpha_n)$, $|\alpha_i| > 1$.

REMARK: Taking an intersection of V with two complementary flags of Hopf submanifolds in H, **we obtain at least 2 elliptic curves.**