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Contact manifolds.

**Definition:** Let $M$ be a smooth manifold, $\dim M = 2n - 1$, and $\omega$ a symplectic form on $M \times \mathbb{R}^+$. Suppose that $\omega$ is homogeneous: $\Psi^* \omega = q^2 \omega$, where $\Psi_q(m, t) = (m, qt)$. Then $M$ is called contact.

**Remark:** The contact form on $M$ is defined as $\theta = \omega \updownarrow \vec{T}$, where $\vec{T} = t \frac{d}{dt}$. Then $d\theta = [d, \cdot \updownarrow \vec{T}] \omega = \text{Lie}_{\vec{T}} \omega = \omega$. Therefore, the form $d\theta^{n-1} \wedge \theta = \frac{1}{n} \omega^n \updownarrow \vec{T}$ is non-degenerate on $M \times \{t_0\} \subset M \times \mathbb{R}^+$.

**Remark:** Usually, a contact manifold is defined as a $(2n - 1)$-manifold with 1-form $\theta$ such that $d\theta^{n-1} \wedge \theta$ is nowhere degenerate.

**Example:** An odd-dimensional sphere $S^{2n-1}$ is contact. Indeed, its cone $S^{2n-1} \times \mathbb{R}^+ = \mathbb{R}^{2n} \setminus 0$ has the standard symplectic form $\sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}$ which is obviously homogeneous.

Contact geometry is an odd-dimensional counterpart to symplectic geometry.
Kähler manifolds.

**Definition:** Let \((M, I)\) be a complex manifold, \(\dim_{\mathbb{C}} M = n\), and \(g\) is Riemannian form. Then \(g\) is called Hermitian if \(g(Ix, Iy) = g(x, y)\).

**Remark:** Since \(I^2 = -\text{Id}\), it is equivalent to \(g(Ix, y) = -g(x, Iy)\). The form \(\omega(x, y) := g(x, Iy)\) is skew-symmetric.

**Definition:** The differential form \(\omega\) is called the Hermitian form of \((M, I, g)\).

**Definition:** A complex Hermitian manifold is called Kähler if \(d\omega = 0\).

**Remark:** Kähler manifolds are the main object of complex algebraic geometry (algebraic geometry over \(\mathbb{C}\)). See e.g. Griffiths, Harris, “Principles of Algebraic Geometry”.


Examples of Kähler manifolds.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and $g$ a $U(n+1)$-invariant Riemannian form. It is called Fubini-Study form on $\mathbb{C}P^n$. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$.

Remark: For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_x\mathbb{C}P^n$ is $U(n)$-invariant, hence unique up to a constant.

Claim: Fubini-Study form is Kähler. Indeed, $d\omega \big|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish (invariants of $U(n)$ are known since XIX century).

Corollary: Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler.
Almost complex manifolds.

A differential-geometric way of looking at Kähler manifolds.

**Definition:** An almost complex structure on a manifold $M$ is an operator $I : TM \rightarrow TM$ such that $I^2 = -\text{Id}$. It is called **integrable** if $I$ is induced by a complex structure.

**Theorem:** A Riemannian almost complex Hermitian manifold $(M, I, g)$ is Kähler if and only if $\nabla \omega = 0$, where $\nabla$ is a Levi-Civita connection.

**Remark:** This theorem is difficult (both ways). Integrability of almost complex structures takes some intensive work on PDEs. The implication $d\omega = 0 \Rightarrow \nabla \omega = 0$ is also non-trivial, but essentially linear-algebraic.

**Remark:** One may think of Kähler manifolds as of symplectic manifolds with a Riemannian structure compatible with a symplectic form. Locally, every symplectic manifold admits a Kähler structure (Darboux).


**Sasakian manifolds.**

**Definition:** Let $M$ be a smooth manifold, $\dim M = 2n - 1$, and $(\omega, I)$ a Kaehler structure on $M \times \mathbb{R}^>0$. Suppose that $\omega$ is homogeneous: $\Psi^*_q \omega = q^2 g$, where $\Psi_q(m, t) = (m, qt)$, and $I$ is $\Psi_q$-invariant. Then $M$ is called **Sasakian**, and $M \times \mathbb{R}^>0$ its **Kähler cone**.

Sasakian geometry is an odd-dimensional counterpart to Kähler geometry.

**Remark:** A Sasakian manifold is obviously contact. Indeed, a Sasakian manifold is a contact manifold equipped with a compatible Riemannian metric.

**Example:** An odd-dimensional sphere $S^{2n-1}$ is Sasakian. Indeed, its cone $S^{2n-1} \times \mathbb{R}^>0 = \mathbb{C}^n \setminus 0$ has the standard Kähler form $\sqrt{-1} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$ which is obviously homogeneous.

S. Sasaki, "On differentiable manifolds with certain structures which are closely related to almost contact structure", Tohoku Math. J. 2 (1960), 459-476.
**Quasiregular Sasakian manifolds.**

**Definition:** Given a contact manifold \((M, \theta)\) with a Riemannian structure \(g\), the dual vector field \(\theta^\#\) is called the **Reeb field** of \((M, \theta, g)\).

**Remark:** For any Sasakian manifold, the Reeb field generates a flow of diffeomorphisms acting on \(M\) by contact isometries. This is obvious from the definition, because the Reeb field \(\theta^\# = It\frac{d}{dt}\) acts by holomorphic isometries on the Kähler cone.

**Definition:** A Sasakian manifold \(M\) is called **quasiregular** if all orbits of the Reeb flow are compact. The space of orbits of the Reeb flow is a complex orbifold. **Every quasiregular Sasakian manifold is a total space of \(S^1\)-bundle over a complex orbifold.**

This is easy to see, because the quotient of \(M\) over the Reeb flow is the same as the quotient of \(CM\) over its complexification, generated by \(\theta^\#\) and \(I\theta^\#\).
Examples of Sasakian manifolds.

Example: Let $X \subset \mathbb{C}P^n$ be a complex submanifold, and $CX \subset \mathbb{C}^{n+1}\setminus 0$ the corresponding cone. The cone $CX$ is obviously Kähler and homogeneous, hence the intersection $CX \cap S^{2n-1}$ is Sasakian. This intersection is an $S^1$-bundle over $X$. This construction gives many interesting contact manifolds, including Milnor’s exotic 7-spheres, which happen to be Sasakian.

Remark: A link of a homogeneous singularity is always Sasakian.

Remark: Every quasiregular Sasakian manifold is obtained this way, for some Kähler metric on $\mathbb{C}^{n+1}$.

Remark: All 3-dimensional Sasakian manifolds are quasiregular (H. Geiges, 1997, F. Belgun, 2000).

Remark: Every Sasakian manifold is diffeomorphic to a quasiregular one (Ornea-V., arXiv:math/0306077)
Sasaki-Einstein manifolds

Definition: A Sasaki-Einstein manifold is a Sasakian manifold with $\text{Ric}(M) = \text{const} \ g$, where Ric is its Ricci curvature, and const a constant.

Boyer and Galicki (also: Demailly, Kollar, Boyer, Galicki, Nakamaye) found a way of constructing Sasaki-Einstein metrics on many 5-dimensional and 7-dimensional manifolds, including Milnor’s exotic spheres, using Nadel’s theory of multiplier ideals. These examples are quasiregular.

Cheeger and Tian conjectured that all Sasaki-Einstein manifolds are quasiregular. In 2005, physicists found very exotic counterexamples, constructed “as BPS limits of the Euclideanised Kerr-de Sitter metrics”. using number theory. These manifolds are studied very intensively.
Sasaki-Einstein manifolds: normalized volume

**Definition:** Let \((M, g)\) be a Sasaki-Einstein manifold, \(\dim_{\mathbb{R}} M = 2n - 1\), with \(\text{Ric}(M) = \text{const} \cdot g\). If we multiply \(g\) by a constant, \(\text{Ric}(M)\) does not change, hence \(\text{const}\) can be anything (it is always positive). **Normalized Einstein equation:** \(\text{Ric}(M) = (2n - 2)g\) **Normalized volume:** \(\text{Vol}(M) := \frac{\text{Vol}(M)}{\text{Vol}(S^{2n-1})}\).

**Remark:** If \(M\) is quasiregular, \(\text{Vol}(M)\) is always rational (expressed through \(c_1\) of the corresponding \(S^1\)-bundle).

**Theorem:** (D. Martelli, J. Sparks, S.-T. Yau, hep-th/0603021) \(\text{Vol}(M)\) is an algebraic number.

**Remark:** This was obtained from string physics, then [MSY] found a proof using Duistermaat-Heckman formula.

**Conjecture:** ([MSY]) The degree of \(\text{Vol}(M)\) is is equal to the dimension of the Lie group obtained as a closure of the Reeb action.
**CR-manifolds.**

**Definition:** Let $M$ be a smooth manifold, $B \subset TM$ a sub-bundle in a tangent bundle, and $I : B \rightarrow B$ an endomorphism satisfying $I^2 = -1$. Consider its $\sqrt{-1}$-eigenspace $B^{1,0}(M) \subset B \otimes \mathbb{C} \subset T_{CM} = TM \otimes \mathbb{C}$. Suppose that $[B^{1,0}, B^{1,0}] \subset B^{1,0}$. Then $(B, I)$ is called a **CR-structure** on $M$.

**Example:** A complex manifold is CR, with $B = TM$. Indeed, $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$ is equivalent to integrability of the complex structure (Newlander-Nirenberg).

**Example:** Let $X$ be a complex manifold, and $M \subset X$ a hypersurface. Then $B := \dim_{\mathbb{C}} TM \cap I(TM) = \dim_{\mathbb{C}} X - 1$, hence $	ext{rk} B = n - 1$. Since $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$, $M$ is a **CR-manifold**.

**Definition:** A Frobenius form of a CR-manifold is the tensor $B \otimes B \rightarrow TM/B$ mapping $X, Y$ to the $\Pi_{TM/B}([X,Y])$. It is an obstruction to integrability of the foliation given by $B$. 

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Contact CR-manifolds.

Complex algebraic geometry is a rich source of contact structures.

**Definition:** Let \((M, B, I)\) be a CR-manifold, with \(\text{codim} \, B = 1\). Then \(M\) is called a **contact CR-manifold** if its Frobenius form is non-degenerate.

**Remark:** Since \([B^{1,0}, B^{1,0}] \subset B^{1,0}\) and \([B^{0,1}, B^{0,1}] \subset B^{0,1}\), the Frobenius form is a pairing between \(B^{0,1}\) and \(B^{1,0}\). This means that it is Hermitian.

**Definition:** Let \((M, B, I)\) be a CR-manifold, with \(\text{codim} \, B = 1\). Then \(M\) is called a **strictly pseudoconvex CR-manifold** if its Frobenius form is positive definite everywhere.

**Example:** Let \(h\) be a function on a complex manifold such that \(\bar{\partial} \partial h = \omega\) is a positive definite Hermitian form, and \(X = h^{-1}(c)\) its level set. Then the Frobenius form of \(X\) is equal to \(\omega\big|_X\). In particular, \(X\) is a **strictly pseudoconvex CR-manifold**.
**CR-geometry of Sasakian manifolds.**

**Claim:** Let $M$ be a Sasakian manifold, $CM = M \times \mathbb{R}^{>0}$ its Kähler cone, and $\varphi(m, t) = t$ the projection of $CM$ to $\mathbb{R}^{>0}$. Then $\sqrt{-1} \, \partial \bar{\partial} \varphi = \omega$ is its Kähler form.

**Proof:**

\[
\sqrt{-1} \, \partial \bar{\partial} \varphi = dd^c \varphi = dId\varphi = dIdt = d(\omega \downarrow \frac{d}{dt}) = \omega \text{ as we have already seen.} \]

**Corollary:**

A Sasakian manifold is strictly pseudoconvex as a CR-manifold.

**Question:**

Which strictly pseudoconvex CR-manifolds admit Sasakian structures?

**Answer:** (Ornea, V., arXiv:math/0606136) Let $M$ be a compact, strictly pseudoconvex CR-manifold. Then $M$ admits a Sasakian metric if and only if $M$ admits a proper, transversal CR-holomorphic $S^1$-action.
A Sasakian analogue of Kodaira embedding theorem.

**Theorem:** (Ornea-V., arXiv:math/0609617) Let $M$ be a compact Sasakian manifold. Then $M$ admits a CR-holomorphic embedding to a sphere.

**An idea of a proof:** A cone of Sasakian manifold is a link of a complex singularity (Ornea-V., arXiv:math/0407231). Embedding projectivization of singularity into $\mathbb{C}P^n$, we obtain a contact immersion of $M$ into a sphere. To make it CR-holomorphic, the following theorem was used.

**Theorem:** (Ornea-V.) Let $X \subset M$ be a complex submanifold of a Kaehler manifold $(M, \omega_M)$. Assume that $X$ is equipped with a Kaehler form $\omega_X$, and the cohomology classes of $\omega_X$ and $\omega_M|_X$ are equal. Then $M$ admits a Kähler form $\omega$ such that $\omega|_X = \omega_X$.

“We can extend a Kähler form from a submanifold to an ambient manifold, if cohomology allows”.