Sasakian manifolds

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Contact manifolds.

Definition: Let M be a smooth manifold, $\dim M = 2n-1$, and ω a symplectic form on $M \times \mathbb{R}^{>0}$. Suppose that ω is **homogeneous**: $\Psi_q^*\omega = q^2\omega$, where $\Psi_q(m,t) = (m, qt)$. Then M is called **contact**.

Remark: The contact form on M is defined as $\theta = \omega \lrcorner \vec{T}$, where $\vec{T} = t\frac{d}{dt}$. Then $d\theta = [d, \cdot \lrcorner \vec{T}]\omega = \operatorname{Lie}_{\vec{T}}\omega = \omega$. Therefore, **the form $d\theta^{n-1} \wedge \theta = \frac{1}{n}\omega^n \lrcorner \vec{T}$ is non-degenerate on $M \times \{t_0\} \subset M \times \mathbb{R}^{>0}$.**

Remark: Usually, a contact manifold is defined as a **$(2n-1)$-manifold with 1-form θ such that $d\theta^{n-1} \wedge \theta$ is nowhere degenerate.**

Example: **An odd-dimensional sphere S^{2n-1} is contact.** Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{R}^{2n} \backslash 0$ has the standard symplectic form $\sum_{i=1}^n dx_{2i-1} \wedge dx_{2i}$ which is obviously homogeneous.

Contact geometry is an odd-dimensional counterpart to symplectic geometry

Kähler manifolds.

Definition: Let (M, I) be a complex manifold, $\dim_{\mathbb{C}} M = n$, and g is Riemannian form. Then g is called **Hermitian** if $g(Ix, Iy) = g(x, y)$.

Remark: Since $I^2 = -\operatorname{Id}$, it is equivalent to $g(Ix, y) = -g(x, Iy)$. The form $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

Definition: The differential form ω is called **the Hermitian form of (M, I, g)**.

Definition: A complex Hermitian manifold is called **Kähler** if $d\omega = 0$.

Remark: **Kähler manifolds are the main object of complex algebraic geometry** (algebraic geometry over $\mathbb{C}$). See e.g. Griffiths, Harris, *"Principles of Algebraic Geometry"*.

Examples of Kähler manifolds.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$.

Remark: For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_x\mathbb{C}P^n$ is $U(n)$-invariant, hence unique up to a constant.

Claim: **Fubini-Study form is Kähler.** Indeed, $d\omega\Big|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish (invariants of $U(n)$ are known since XIX century).

Corollary: Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler.

Almost complex manifolds.

A differential-geometric way of looking at Kähler manifolds.

Definition: An almost complex structure on a manifold M is an operator $I: TM \longrightarrow TM$ such that $I^2 = -\operatorname{Id}$. It is called **integrable** if I is induced by a complex structure.

Theorem: A Riemannian almost complex Hermitian manifold (M, I, g) is **Kähler if and only if $\nabla\omega = 0$,** where ∇ is a Levi-Civita connection.

Remark: This theorem is difficult (both ways). Integrability of almost complex structures takes some intensive work on PDEs. The implication $d\omega = 0 \Rightarrow \nabla\omega = 0$ is also non-trivial, but essentially linear-algebraic.

Remark: One may think of Kähler manifolds as of symplectic manifolds with a Riemannian structure compatible with a symplectic form. Locally, every symplectic manifold admits a Kähler structure (Darboux).

Sasakian manifolds.

Definition: Let M be a smooth manifold, $\dim M = 2n-1$, and (ω, I) a Kaehler structure on $M \times \mathbb{R}^{>0}$. Suppose that ω is **homogeneous**: $\Psi_q^*\omega = q^2 g$, where $\Psi_q(m,t) = (m, qt)$, and I is Ψ_q-invariant. Then M is called **Sasakian**, and $M \times \mathbb{R}^{>0}$ its **Kähler cone**.

Sasakian geometry is an odd-dimensional counterpart to Kähler geometry

Remark: A Sasakian manifold is obviosly contact. Indeed, **a Sasakian manifold is a contact manifold equipped with a compatible Riemannian metric.**

Example: **An odd-dimensional sphere S^{2n-1} is Sasakian.** Indeed, its cone $S^{2n-1} \times \mathbb{R}^{>0} = \mathbb{C}^n \backslash 0$ has the standard Kähler form $\sqrt{-1}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$ which is obviously homogeneous.

S. Sasaki, "On differentiable manifolds with certain structures which are closely related to almost contact structure", Tohoku Math. J. 2 (1960), 459-476.

Quasiregular Sasakian manifolds.

Definition: Given a contact manifold (M, θ) with a Riemannian structure g, the dual vector field $\theta^\sharp$ is called **the Reeb field** of (M, θ, g).

Remark: For any Sasakian manifold, **the Reeb field generates a flow of diffeomorphisms acting on M by contact isometries**. This is obvious from the definition, because the Reeb field $\theta^\sharp = It\frac{d}{dt}$ acts by holomorphic isometries on the Kähler cone.

Definition: A Sasakian manifold M is called **quasiregular** if all orbits of the Reeb flow are compact. The space of orbits of the Reeb flow is a complex orbifold. **Every quasiregular Sasakian manifold is a total space of S^1-bundle over a complex orbifold.**

This is easy to see, because the quotient of M over the Reeb flow is the same as the quotient of CM over its complexification, generated by $\theta^\sharp$ and $I\theta^\sharp$.

Examples of Sasakian manifolds.

Example: Let $X \subset \mathbb{C}P^n$ be a complex submanifold, and $CX \subset \mathbb{C}^{n+1}\backslash 0$ the corresponding cone. The cone CX is obviously Kähler and homogeneous, hence **the intersection $CX \cap S^{2n-1}$ is Sasakian.** This intersection is an S^1-bundle over X. This construction gives many interesting contact manifolds, including Milnor's exotic 7-spheres, which happen to be Sasakian.

Remark: A link of a homogeneous singularity is always Sasakian.

Remark: Every quasiregular Sasakian manifold is obtained this way, for some Kähler metric on $\mathbb{C}^{n+1}$.

Remark: All 3-dimensional Sasakian manifolds are quasiregular (H. Geiges, 1997, F. Belgun, 2000).

Remark: **Every Sasakian manifold is diffeomorphic to a quasiregular one** (Ornea-V., arXiv:math/0306077)

Sasaki-Einstein manifolds

Definition: A **Sasaki-Einstein manifold** is a Sasakian manifold with $\operatorname{Ric}(M) = \operatorname{const} g$, where Ric is its Ricci curvature, and const a constant.

Boyer and Galicki (also: Demailly, Kollar, Boyer, Galicki, Nakamaye) **found a way of constructing Sasaki-Einstein metrics on many 5-dimensional and 7-dimensional manifolds,** including Milnor's exotic spheres, using Nadel's theory of multiplier ideals. These examples are quasiregular.

Cheeger and Tian conjectured that all Sasaki-Einstein manifolds are quasiregular. In 2005, physicists found very exotic counteraxamples, constructed "as BPS limits of the Euclideanised Kerr-de Sitter metrics". using number theory. These manifolds are studied very intensively.

Sasaki-Einstein manifolds: normalized volume

Definition: Let (M, g) be a Sasaki-Einstein manifold, $\dim_{\mathbb{R}} M = 2n - 1$, with $\operatorname{Ric}(M) = \operatorname{const} g$. If we multiply g by a constant, $\operatorname{Ric}(M)$ does not change, hence const can be anything (it is always positive). **Normalized Einstein equation**: $\operatorname{Ric}(M) = (2n - 2)g$ **Normalized volume:** $\operatorname{Vol}(M) := \frac{\operatorname{Vol}(M)}{\operatorname{Vol}(S^{2n-1})}$.

Remark: If M is quasiregular, $\operatorname{Vol}(M)$ is always rational (expressed through c_1 of the corresponding S^1-bundle).

Theorem: (D. Martelli, J. Sparks, S.-T. Yau, hep-th/0603021) **$\operatorname{Vol}(M)$ is an algebraic number.**

Remark: This was obtained from string physics, then [MSY] found a proof using Duistermaat-Heckman formula.

Conjecture: ([MSY]) The degree of $\operatorname{Vol}(M)$ is is equal to the dimension of the Lie group obtained as a closure of the Reeb action.

CR-manifolds.

Definition: Let M be a smooth manifold, $B \subset TM$ a sub-bundle in a tangent bundle, and $I: B \longrightarrow B$ an endomorphism satisfying $I^2 = -1$. Consider its $\sqrt{-1}$-eigenspace $B^{1,0}(M) \subset B \otimes \mathbb{C} \subset T_{\mathbb{C}}M = TM \otimes \mathbb{C}$. Suppose that $[B^{1,0}, B^{1,0}] \subset B^{1,0}$. Then (B, I) is called **a CR-structure on** M.

Example: A complex manifold is CR, with $B = TM$. Indeed, $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$ **is equivalent to integrability of the complex structure** (Newlander-Nirenberg).

Example: Let X be a complex manifold, and $M \subset X$ a hypersurface. Then $B := \dim_{\mathbb{C}} TM \cap I(TM) = \dim_{\mathbb{C}} X - 1$, hence $\operatorname{rk} B = n - 1$. Since $[T^{1,0}X, T^{1,0}X] \subset T^{1,0}X$, M **is a CR-manifold.**

Definition: A Frobenius form of a CR-manifold is the tensor $B \otimes B \longrightarrow TM/B$ mapping X, Y to the $\Pi_{TM/B}([X, Y])$. It is an obstruction to integrability of the foliation given by B.

Contact CR-manifolds.

Complex algebraic geometry is a rich source of contact structures.

Definition: Let (M, B, I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a contact CR-manifold** if its Frobenius form is non-degenerate.

Remark: Since $[B^{1,0}, B^{1,0}] \subset B^{1,0}$ and $[B^{0,1}, B^{0,1}] \subset B^{0,1}$, the Frobenius form is a pairing between $B^{0,1}$ and $B^{1,0}$. This means that it is Hermitian.

Definition: Let (M, B, I) be a CR-manifold, with $\operatorname{codim} B = 1$. Then M is called **a strictly pseudoconvex CR-manifold** if its Frobenius form is positive definite everywhere.

Example: Let h be a function on a complex manifold such that $\partial\bar{\partial}h = \omega$ is a positive definite Hermitian form, and $X = h^{-1}(c)$ its level set. Then the Frobenius form of X is equal to $\omega\big|_X$. In particular, X **is a strictly pseudoconvex CR-manifold**.

CR-geometry of Sasakian manifolds.

Claim: Let M be a Sasakian manifold, $CM = M \times \mathbb{R}^{>0}$ its Kähler cone, and $\varphi(m,t) = t$ the projection of CM to $\mathbb{R}^{>0}$. Then $\sqrt{-1}\,\partial\bar\partial\varphi = \omega$ **is its Kähler form.**

Proof:
$\sqrt{-1}\,\partial\bar\partial\varphi = dd^c\varphi = dId\varphi = dIdt = d(\omega \lrcorner \frac{d}{dt}) = \omega$ as we have already seen. ■

Corollary:
A Sasakian manifold is strictly pseudoconvex as a CR-manifold.

Question:
Which strictly pseudoconvex CR-manifolds admit Sasakian structures?

Answer: (Ornea, V., arXiv:math/0606136) Let M be a compact, strictly pseudoconvex CR-manifold. **Then M admits a Sasakian metric if and only if M admits a proper, transversal CR-holomorphic S^1-action.**

A Sasakian analogue of Kodaira embedding theorem.

Theorem: (Ornea-V., arXiv:math/0609617) **Let M be a compact Sasakian manifold. Then M admits a CR-holomorphic embedding to a sphere.**

An idea of a proof: A cone of Sasakian manifold is a link of a complex singularity (Ornea-V., arXiv:math/0407231). Embedding projectivization of singularity into $\mathbb{C}P^n$, we obtain a contact immersion of M into a sphere. To make it CR-holomorphic, the following theorem was used.

Theorem: (Ornea-V.) Let $X \subset M$ be a complex submanifold of a Kaehler manifold (M, ω_M). Assume that X is equipped with a Kaehler form ω_X, and the cohomology classes of ω_X and $\omega_M\Big|_X$ are equal. **Then M admits a Kähler form ω such that $\omega\Big|_X = \omega_X$.**

"We can extend a Kähler form from a submanifold to an ambient manifold, if cohomology allows".