Shelukhin's quasimorphisms

Misha Verbitsky

Geometric structures on manifolds

IMPA, August 31, 2023

Quasimorphisms

DEFINITION: A **quasimorphism** is a map q from a group $\Gamma \to \mathbb{R}$ such that $q(x) = -q(x^{-1}$ and $|q(xy) - q(y) - q(x)| < C$, for some fixed constant $C > 0$.

EXAMPLE: (Brooks quasimorphism)
Let $\mathbb{F}_n$ be a free group with generators $z_1, ..., z_n$. Non-trivial elements of $\mathbb{F}_n$ are represented by **reduced words**, that is, sequences of the letters $z_1, ..., z_n, z_1^{-1}, ..., z_n^{-1}$ such that z_i and z_i^{-1} don't appear next to each other. Fix a reduced word W. Given a reduced word W_1, let $n_+(W_1)$ be the maximal number of disjoint subsequences equal to W in W_1, $n_-(W_1)$ be the maximal number of disjoint subsequences equal to W^{-1} in W_1, and $n(W_1) := n_+(W_1) - n_-(W_1)$.

EXERCISE: Prove that $|n(W_1W_2) - n(W_1) - n(W_2)| \leqslant 2$. In other words, n is a quasimorphism. It is called **the Brooks quasimorphism**. It was defined by Brooks and Gromov (1978).

Equivalent quasimorphisms

DEFINITION: Two quasimorphisms $q_1, q_2 : \Gamma \to \mathbb{R}$ are called **equivalent** if $|q_1(g) - q_2(g)| < C$ for some fixed constant C. A quasimorphism $q\ \Gamma \to \mathbb{R}$ is called **homogeneous** if $q(z^n) = nq(z)$ for any $z \in \Gamma$ and and any $n \in \mathbb{Z}$.

PROPOSITION: **Any quasimorphism is equivalent to a unique homogeneous quasimorphism.**

Proof: Left as an exercise. ■

Quasimorphisms and stable commutator length

DEFINITION: A group G is called **perfect** if $G = [G, G]$. **The commutator length** of $g \in G$ is the minimal number m such that g can be represented as a product of m commutators. **The stable commutator length** of an element $g \in G$ is defined as $\operatorname{scl}(g) := \lim_{n\to\infty} \frac{\operatorname{cl}(g^n)}{n}$, where cl is the commutator length.

THEOREM: (Pasiencier and Wang)
Every element of a connected semisimple complex Lie group is a commutator.

THEOREM: (Dragomir Z. Đoković)
The commutator length of any element in a semisimple real algebraic Lie group **is 1 or 2.** It is equal to 2 **for some elements in $SU(p, q)$.**

LEMMA: (Bavard)
Let q be a homogeneous quasimorphism. Then $q(xyx^{-1}y^{-1}) = 0$.

In particular, **any homogeneous quasimorphism on a group with finite stable commutator length vanishes.** The converse is also true!

THEOREM: (Bavard duality)
For any g in a perfect group G, **g has non-zero stable commutator length if and only if there exists a homogeneous quasimorphism $q: G \longrightarrow \mathbb{R}$ such that $q(g) \neq 0$.**

The Maslov index

DEFINITION: Let $V=\mathbb{R}^{2n},\omega$ be a symplectic vector space, and $\mathrm{Gr}_L(V)$ the **Lagrangian Grassmannian**, that is, the Grassmannian of all Lagrangian subspaces.

PROPOSITION: $\pi_1(\mathrm{Gr}_L(V))=\mathbb{Z}$.

Proof. Step 1: Consider a complex structure on V compatible with the symplectic structure, and fix a non-degenerate complex volume form $\Phi\in\Lambda^{n,0}(V)$, $|\Phi|=1$. Consier a Lagrangian subspace $W\subset V$, and let Vol_W be its Euclidean volume form. The restriction $\Phi|_W$ is a complex-valued form which satisfies $\left|\frac{\Phi}{\mathrm{Vol}_W}\right|=1$. The number $\mathrm{ph}(W):=\frac{\Phi}{\mathrm{Vol}_W}$ is called **the phase** of W, we consider $\mathrm{ph}(W)$ as an element in $U(1)\subset\mathbb{C}$.

Step 2: The Maslov index is a homomorphism $\pi_1(\mathrm{Gr}_L(V))\longrightarrow\mathbb{Z}$ induced by the phase map $\mathrm{ph}:\ \mathrm{Gr}_L(V))\longrightarrow U(1)$. To prove that it induces an isomorphism on π_1, we notice that $U(V)$ acts on $\mathrm{Gr}_L(V)$ transitively, with $O(W)$ stabiliser of a point $W\in\mathrm{Gr}_L(V)$, giving an isomorphism $\mathrm{Gr}_L(V)=\frac{U(V)}{O(W)}$ and an exact sequence $\pi_1(O(W))\longrightarrow\pi_1(U(V))\longrightarrow\pi_1(\mathrm{Gr}_L(V))\longrightarrow\pi_0(O(W))$. Since $\pi_1(U(n))=\mathbb{Z}$, and rotation of the phase induces the change of orientation in W, this implies $\pi_1(\mathrm{Gr}_L(V))=\mathbb{Z}$. ■

The metaplectic group

PROPOSITION: In these assumptions, $\pi_1(\mathrm{Sp}(V)) = \mathbb{Z}$ **and the natural action map** $\mathrm{Sp}(V) \longrightarrow \mathrm{Gr}_L(V))$ **induces an isomorphism on** π_1**.**

Proof: Polar decomposition map gives a diffeomorphism $\mathrm{Sp}(V) = A \times U(V)$, where A is the space of symmetric positive definite matrices; since A is contractible, we obtain that $\mathrm{Sp}(V)$ is homotopy equivalent to $U(V)$. This gives $\pi_1(\mathrm{Sp}(V)) = \mathbb{Z}$. Since the generator of $\pi_1(\mathrm{Gr}_L(V))$ is induced by rotations, the map $\mathrm{Sp}(V) \longrightarrow \mathrm{Gr}_L(V))$ induces an isomorphism of the fundamental groups. ■

DEFINITION: **The metaplectic group** $\tilde{\mathrm{Sp}}(V)$ is the universal covering of $\mathrm{Sp}(V)$.

REMARK: Any finite-dimensional representation of $\tilde{\mathrm{Sp}}(V)$ is factorized through $\mathrm{Sp}(V)$. The group $\tilde{\mathrm{Sp}}(V)$ faithfully acts on a Hilbert space; the corresponding representation is called **the Weil representation**. It is defined in the same way as the spinorial representation, with **the Weyl algebra** (the algebra of polynomial differential operators) in place of the Clifford algebra.

(both H. Weyl and A. Weil are mentioned here, and it is not a misprint).

The Maslov quasimorphism

DEFINITION: Let $u: [a,b] \longrightarrow U(1)$ be a continuous map, and $\tilde{u}$ its lifting to the universal covering $\mathbb{R}$ of $U(1) = \frac{\mathbb{R}}{2\pi\mathbb{Z}}$. **The index** of u is $\tilde{u}(b) - \tilde{u}(a)$.

DEFINITION: Let $x \in \tilde{\mathrm{Sp}}(V)$ be an element in the metaplectic group. Since $\tilde{\mathrm{Sp}}(V)$ is simply connected, there exists a unique up to homotopy path x_t : $[0,1] \longrightarrow \mathrm{Sp}(V)$ connecting x to unity. Fix $W \in \mathrm{Gr}_L(V)$, and **let $q(x)$ be the index of the map $t \mapsto \mathrm{ph}(x_t(W))$.**

CLAIM: The map q is a quasimorphism, which is equal to the Maslov index of the path x_t connecting two preimages of the unity.

Proof: Left as an exercise.

DEFINITION: This map is called **the Maslov quasimorphism.**

The Entov quasimorphism and its generalization by Shelukhin

Let M be a compact symplectic manifold with $c_1(M) = 0$, and $\widetilde{\text{Symp}}(M)$ the universal cover of the connected component its symplectomorphism group. In 2004, M. Entov constructed a quasimorphism $q: \widetilde{\text{Symp}}(M) \longrightarrow \mathbb{R}$ **by taking a path γ_t connecting $\gamma \in \text{Symp}(M)$ to unity, computing the Maslov indices of the paths $\gamma_t(m)$ for all $m \in M$ and integrating over M.**

This construction was generalized by E. Shelukhin using Donaldson's construction of the moment map on the space of compatible almost complex structures.

The group of symplectic diffeomorphism acts on the space of compatible complex structures **(which is infinite-dimensional and Kähler)** by holomorphic isometries; its moment map was computed by Donaldson.

Shelukhin constructed a generalization of Entov's quasimorphism using this moment map. For this purpose he defined a general formalism which can be applied to symplectic manifolds equipped with a certain geometric structure, called **"a Domic-Toledo structure";** this structure follows from hyperbolicity, but is less restrictive.

The space of compatible almost complex structures

DEFINITION: Let (V,ω) be a symplectic vector space, and $I\in \operatorname{End}(V)$ a linear operator which satisfies $I^2=-\operatorname{Id}$, We say that I **is a compatible complex structure** if $\omega(\cdot, I\cdot)$ is a positive definite symmetric form.

REMARK: Let $\mathcal{M}$ be the space of compatible complex structures on (V,ω), and $V_{\mathbb C}:= V\otimes_{\mathbb R}{\mathbb C}$. A complex structure I on V can be obtained by taking a subspace $V^{1,0}\subset V_{\mathbb C}$ such that $V_{\mathbb C}= V^{1,0}\oplus \overline{V^{1,0}}$. Then I is compatible with ω if and only if $V^{1,0}$ is Lagrangian and satisfies $\sqrt{-1}\,\omega(x,\overline{x})>0$ for any non-zero $x\in V^{1,0}$. The second condition is open, hence $\mathcal{M}$ is an open subset in a complex Lagrangian Grassmanian $\operatorname{Gr}_L(V_{\mathbb C})$. The tangent space $T_W \operatorname{Gr}_L(V_{\mathbb C})$ is the space of all maps $A\in \operatorname{Hom}(W, V/W)$ which satisfy $A(x,y)=-A(y,x)$; therefore, $\mathcal{M}$ **admits an invariant complex structure.**

The Siegel domain

REMARK: The space $\mathcal{M}$ of compatible complex structures is clearly isomorphic to $\mathsf{Sp}(n)/U(n)$; this is a symmetric space of negative curvature, equipped with an $\mathsf{Sp}(n)$-invariant complex structure. Since $\mathcal{M}$ is symmetric, it follows that $\mathsf{Sp}(n)/U(n)$ is equipped with a natural $\mathsf{Sp}(n)$-invariant complex structure, which is integrable and Kähler because **any almost complex Hermitian symmetric space is actually Kähler.** The space $\mathsf{Sp}(n)/U(n)$ was introduced by I. Pyatetsky-Shapiro in 1959 under the name "Siegel domains". Now these spaces are known as **Siegel domains**, or **Pyatetsky-Shapiro domains**.

Also, $\mathsf{Sp}(n)/U(n)$ appears in E. Cartan's classification of Hermitian symmetric spaces as "III_n" (1935).

The Siegel domain **is a Teichmüller space of Abelian varieties,** that is, **the universal covering of the corresponding moduli space.**

The space of compatible almost complex structures on a manifold

Let (M,ω) be a symplectic manifold. $\dim_{\mathbb{R}} M = 2n$. Denote by $\mathcal{G}$ the principal $\mathsf{Sp}(n)$-bundle of all symplectic frames on M, and let $\mathcal{M}(M)$ be the space of all compatible almost complex structures on M. Clearly, $\mathcal{M}(M) = \mathcal{G} \times_{\mathsf{Sp}(n)} \mathcal{M}$, where $\mathcal{M} = \mathsf{Sp}(n)/U(n)$ is the space of all compatible almost complex structures on $V = \mathbb{R}^{2n}$. The projection $\mathcal{M}(M) \longrightarrow M$ is a locally trivial map with fiber $\mathcal{M}$. **Compatible almost complex structure on M bijectively correspond to sections of this fibration.**

CLAIM: Let (M,ω) be a symplectic manifold, and $\pi: \mathcal{M}(M) \longrightarrow M$ the $\mathsf{Sp}(n)/U(n)$-fibration constucted above. Denote the space of sections of π by $\mathcal{C}_M$; this space is identified with the space of induced complex structures, as show above. Since the fibers of π are complex manifolds, the space $\mathcal{C}_M$ acquires a natural almost complex structure. Then **this complex structure is integrable**.

Proof: Darboux theorem implies that locally in M, the projection $\mathcal{M}(M) \longrightarrow M$ is isomorphic to $\mathcal{M} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and this diffeomorphism is compatible with the complex structure on fibers of this projection. ■

The space of almost complex structures is symplectic

REMARK: The total space $\mathcal{M}(M)$ of the fibration $\mathcal{M}(M) \longrightarrow M$ **is equipped with a natural symplectic structure as follows:** locally in M it is a product of two symplectic manifolds M and $\mathcal{M}$, and the transition functions are symplectomorphisms.

DEFINITION: We define a non-degenerate 2-form on $\mathcal{C}_M$ as follows. Given two vectors $u, v \in T_I\mathcal{C}_M$, we interpret u, v as a collection of vectors tangent to the fibers of π. These fibers are identified with $\mathcal{M} = \mathsf{Sp}(n)/U(n)$, hence are symplectic. We use this symplectic structure to pair v with u at each point of M, and integrate the resulting function on M using the volume form of ω as a measure.

CLAIM: The antisymmetric 2-form ω_C on $\mathcal{C}_M$ defined above is closed, hence defines a symplectic and Kähler structure on $\mathcal{C}_M$.

Proof: This is clear locally in M, hence $d\omega = 0$ on vector fields with support in a Darboux chart of M. Since any vector field can be obtained as a linear combination of those, we have $d\omega_C = 0$. ■

The moment map

DEFINITION: Let G be a Lie group acting on a symplectic manifold (M, ω) by symplectomorphisms. Consider $v \in \operatorname{Lie}(G)$ as a vector field on M. Then $\operatorname{Lie}_v \omega = 0$, and Cartan formula gives $d(i_v\omega) = 0$. **The Hamiltonian** of v is a function H_v on M such that $dH_v = i_v\omega$. A vector field $v \in TM$ is called **Hamiltonian** if the 1-form $i_v\omega$ is exact.

DEFINITION: A Lie group action is **Hamiltonian** if any vector field $v \in \operatorname{Lie} G$ is Hamiltonian; the corresponding **moment map** takes vector field $v \in \operatorname{Lie} G$ and produces a function on M. We interpret it as a map $\mu: M \longrightarrow \mathfrak{g}^*$, where $\mathfrak{g} = \operatorname{Lie} G$ and $\mathfrak{g}^*$ is the dual space.

REMARK: Consider the Lie algebra Ham_M of Hamiltonian vector fields on M. We identify M with the space $C^\infty M/\mathbb{R}$ of functions up to a constant. For any Hamiltonian action of Ham_M on a symplectic manifold X, the corresponding moment map is a map $X \longrightarrow (C^\infty M/\mathbb{R})^*$. There is a perfect bilinear symmetric pairing $C^\infty M \times C^\infty M \longrightarrow \mathbb{R}$, taking f, g to $\int_M fg \operatorname{Vol}_\omega$. **In the next theorem, a map $\mathcal{C}_M \longrightarrow C^\infty M$ is interpreted as a moment map using this pairing.**

The moment map on the space of almost complex structures

THEOREM: Let (M,ω) be a compact symplectic manifold, and $\mathcal{C}_M$ the space of all compatible almost complex structures constructed above. By construction, any symplectomorphism of M acts on $\mathcal{C}_M$ preserving the Kähler structure. **This action is Hamiltionian, and the corresponding moment map takes the complex structure $I \in \mathcal{C}_M$ to the function $S(I) := \frac{R\wedge\omega^{n-1}}{\omega^n}$** where R is the curvature of the connection on the canonical bundle induced by the Bismut connection on the Hermitian manifold (M,I,ω).

Proof: S. K Donaldson, *Remarks on gauge theory, complex geometry and 4-manifold topology, Fields Medallists' lectures,* World Sci. Ser. 20th Century Math., vol. 5, World Sci. Publ., River Edge, NJ, 1997, pp. 384-403. ■

REMARK: **$S(I)$ is the scalar curvature of the Riemannian metric on M induced by I**, if I is integrable.

DEFINITION: The volume map $\mu: M \longrightarrow \mathfrak{g}^*$ is called **equivariant** if it exchanges the G-action on M given originally with the coadjoint action on $\mathfrak{g}^*$, that is, if for all $g\in G$ one has $\mu(gm) = \operatorname{coad}(g)\mu m$.

CLAIM: The moment map constructed by Donaldson **is clearly** Ham_M-**equivariant.** ■

Domic-Toledo structure

DEFINITION: Let (M,ω) be a symplectic manifold. A **Domic-Toledo structure on M** is a collection of paths $K(x,y)$ connecting x to y for any pair of points $x,y\in M$, such that any triangle formed by $x,y,z\in M$ can be filled by a disk D, and $|\int_D\omega|$ is bounded by a constant C independent from the choice of x,y,z and the paths.

We always parametrize the paths γ in $K(x,y)$ by $[0,1]$.

EXAMPLE: Let M be a universal covering of a compact symplectic manifold admitting a metric of negative sectional curvature, and $K(x,y)$ the geodesic connecting x to y. Since a geodesic triangle in a Riemannian manifold with negative sectional curvature $k<-\varepsilon<0$ has bounded area A, we obtain $|\int_D\omega|<C$, where C is A times the maximal eigenvalue of ωg^{-1}. Therefore, **$(M,\omega,K(x,y))$ is a Domic-Toledo space.**

DEFINITION: For this reason, the paths $\gamma\in K(x,y)$ are called **geodesic paths** (associated with the Domic-Toledo structure), and the corresponding triangles **the geodesic triangles**.

Domic-Toledo structure on the space of complex structures

EXAMPLE: A Hermitian symmetric space of negative curvature (such as the Siegel space) **is Domic-Toledo,** for the same reason.

EXAMPLE: Let (M, ω) be a compact symplectic manifold, and $\mathcal{C}_M$ the space of almost complex structures, considered as a symplectic manifold. Consider two points $a, b \in \mathcal{C}_M$ as sections of the bundle $\pi: \mathcal{M}(M) \longrightarrow M$. The fibers of this bundle are Hermitian symmetric spaces of negative curvature, hence any two points $x, y \in \mathcal{M} = \mathrm{Sp}(n)/U(n)$ can be connected by a unique geodesic, which depends smoothly on x, y. We connect a to b by geodesics in each fiber of π; this gives a path $K(a, b)$ in $\mathcal{C}_M$. The symplectic volume of a disc filling the geodesic triangle is integral over M of discs which fill the corresponding triangles in the fibers of π, **hence its volume is bounded.**

COROLLARY: Let (M, ω) be a compact symplectic manifold, and $\mathcal{C}_M$ the space of almost complex structures, considered as a symplectic manifold. **Then the fiberwise geodesics $K(a, b)$ define a Domic-Toledo structure on $\mathcal{C}_M$.**

The action functional

DEFINITION: (Weinstein, 1992)
Let (M, ω) be a simply connected symplectic manifold, equipped with a Hamiltonian action of a Lie group G. For symplicity, we assume that (M, ω) is **aspherical**, that is, $\langle \omega, \pi_2(M) \rangle = 0$. Consider a loop $\gamma_t \in G$, let $x \in M$ and let D be a disk on M with boundary in γ_t. Define **the action functional** $A(\gamma) = \int_D \omega - \int_0^1 H_t dt$, where $H_t := \mu(\gamma_t')$ is the Hamiltonian of the vector field $\gamma_t' \in \operatorname{Lie}(G)$.

THEOREM: The action functional defines a homomorphism $\pi_1(G) \to \mathbb{R}$; **it is independent from the choice of γ_t in its homotopy class and from the choice of $x \in M$.**

Proof: A. Weinstein, Cohomology of symplectomorphism groups and critical values of Hamiltonians, Math. Z. 201 (1989), no. 1, 75-82. ∎

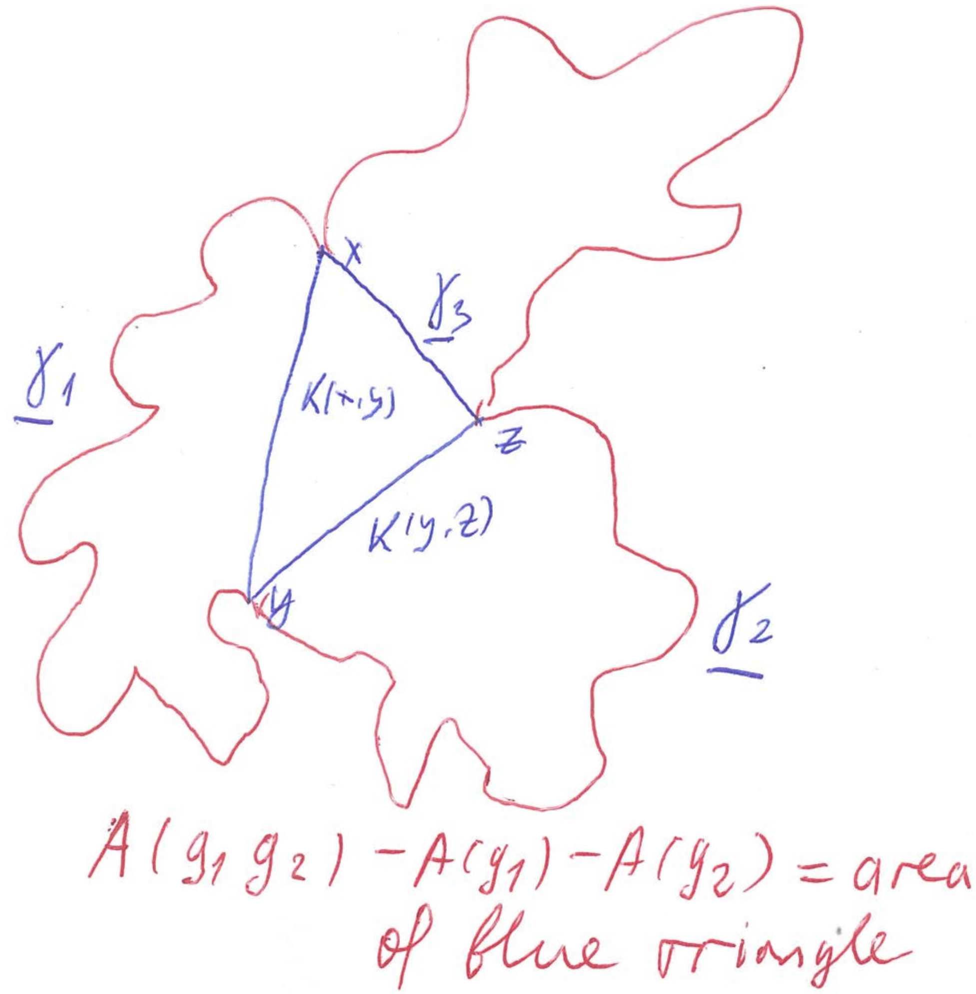
To obtain a quasimorphism, we generalize this construction, using paths instead of closed loops.

Domic-Toledo structure and the action functional

THEOREM: (Shelukhin, 2012)
Let G be a simply connected, connected Lie group (possibly infinite-dimensional) acting on a Domic-Toledo space $(X, \omega, K(x,y))$. Assume that this action is Hamiltonian on (X, ω) and preserves the path sets $K(x,y)$, and admits an equivariant moment map $\mu: X \longrightarrow \mathfrak{g}^*$. Starting from a point $g \in \tilde{G}$, let $\underline{\gamma}$ be a path connecting g to the unity. For any $x \in M$, we consider a closed loop γ by gluing to $\underline{\gamma}$ the path $K(g(x), x)$. **Then the action functional $A_x(g) = \int_D \omega - \int_0^1 H_t dt$, defines a quasimorphism $G \to \mathbb{R}$.** Moreover, **this quasimorphism is independent from the choice of the path γ, and for any $x, y \in X$, the quasimorphisms A_x, A_y are equivalent.**

Domic-Toledo structure and the action functional (2)

Proof:



■

Formally Kähler manifolds

DEFINITION: Let F be a Fréchet manifold. The **sheaf of vector fields** TF on F is a sheaf of continuous derivations of its structure sheaf.

REMARK: A commutator of two derivations is again a derivation. Therefore, **TF is a sheaf of Lie algebras.**

DEFINITION: Let F be a Fréchet manifold, and $I: TF \longrightarrow TF$ a smooth $C^\infty F$-linear endomorphism of the tangent bundle satisfying $I^2 = -1$. Then I is called **an almost complex structure on F**.

REMARK: Clearly, I defines a decomposition $TF \otimes \mathbb{C} = T^{1,0}F \oplus T^{0,1}F$, where $T^{1,0}F$ is the $\sqrt{-1}$-eigenspace of I, and $T^{0,1}F$ the $-\sqrt{-1}$-eigenspace. Indeed, $x = \frac{1}{2}(x + \sqrt{-1}\, Ix) + \frac{1}{2}(x - \sqrt{-1}\, Ix)$.

DEFINITION: An almost complex structure on a Fréchet manifold (F, I) is called **formally integrable**, if $[T^{1,0}F, T^{1,0}F] \subset T^{1,0}F$,

DEFINITION: Let (F, I) be a formally integrable almost complex Fréchet manifold, g a Hermitian structure on F, and ω be the corresponding $(1,1)$-form. We say that (F, I, g) is **formally Kähler** if ω is closed.

Formally Kähler structure on the knot space

DEFINITION: A knot on M is a non-parametrized, immersed free loop. Note that we allow self-intersecting knots, though for technical reasons it's better to exclude loops going around the same knot several times (otherwise the knot space becomes an orbifold, instead of a manifold).

DEFINITION: Let $\operatorname{Knot}(M)$ be the space of knots on an oriented Riemannian 3-manifold M. For each $S \in \operatorname{Knot}(M)$, $T_S \operatorname{Knot}(M)$ is the space of sections of a normal bundle NS. Let γ be a unit tangent vector to S. **The vector product with γ defines a complex structure on the vector space NS.**

THEOREM: (Brylinski)
This complex structure is formally integrable. Moreover, **the corresponding metric on** $\operatorname{Knot}(M)$ **is formally Kähler.**

Symplectic structure on the knot space

DEFINITION: Let $\operatorname{Knot}^m(M) \subset \operatorname{Knot}(M) \times M$ be the space of marked knots, that is, pairs $(S^1 \stackrel{\gamma}{\hookrightarrow} \operatorname{Knot}(M), s \in S^1)$, where $|\gamma'| = const$. Clearly, the forgetful map $\operatorname{Knot}^m(M) \stackrel{\pi}{\longrightarrow} \operatorname{Knot}(M)$ is an S^1-fibration. The fiberwise integration map

$$\Lambda^i(\operatorname{Knot}^m(M)) \stackrel{\pi_*}{\longrightarrow} \Lambda^{i-1}(\operatorname{Knot}(M))$$

is defined as usual,

$$\pi_*(\alpha)|_S := \int_{\pi^{-1}(S)} \left(\alpha \lrcorner \frac{d}{dt}\right) dt$$

where t is a parameter on S.

REMARK: The pushforward map π_* **always commutes with the de Rham differential.** This gives an interesting map

$$\pi_*\sigma^* : \ \Lambda^i(M) \longrightarrow \Lambda^{i-1}(\operatorname{Knot}(M))$$

commuting with the de Rham differential.

CLAIM: Let M be a 3-manifold, and ρ a volume form on M. **Then the form $\pi_*\sigma^*(\rho)$ is symplectic.** It is equal to Brylinski's Kähler form defined above.