Singular hyperkähler varieties

Misha Verbitsky

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IMPA

Real analytic varieties

DEFINITION: Let $B \subset \mathbb{R}^n$ be an open ball, equipped with its sheaf of real analytic functions. A **weak real analytic space** is a ringed space which is locally isomorphic to $Spec(\mathcal{O}(B)/I)$, for some ideal $I \subset \mathcal{O}(B)$. A **real analytic variety** is a weak real analytic space without nilpotents in its structure sheaf.

REMARK: In the literature, a **real analytic space** is a weak real analytic space for which the structure sheaf is coherent (i. e., locally finitely generated and finitely presentable). We will not use this notion.

Real structures on complex varieties

DEFINITION: A smooth map $\Psi: M \longrightarrow N$ on an almost complex variety (M, I) is called **antiholomorphic** if $d\Psi(I) = -I$. A function f is called **antiholomorphic** if $\overline{f}$ is holomorphic.

EXERCISE: Prove that **an antiholomorphic function on M defines an antiholomorphic map from M to $\mathbb{C}$.**

EXERCISE: Prove that a map $\Psi: M \longrightarrow N$ of almost complex manifolds is antiholomorphic **if and only if $\Psi^*(\Lambda^{0,1}(N)) \subset \Lambda^{1,0}(M)$.**

EXERCISE: Let ι be a map from a complex variety M to itself. Prove that **ι is antiholomorphic if and only if $\iota^*(f)$ is antiholomorphic for any holomorphic function f on $U \subset M$.**

DEFINITION: A real structure on a complex manifold M is an antiholomorphic involution $\tau: M \longrightarrow M$.

EXAMPLE: Complex conjugation defines a real structure on $\mathbb{C}^n$.

Fixed points of real structures on manifolds

PROPOSITION: Let M be a complex manifold and $\iota: M \longrightarrow M$ a real structure. Denote by M^ι the fixed point set of ι. Then, **for each $x \in M^\iota$ there exists a ι-invariant coordinate neighbourhood with holomorphic coordinates $z_1, ..., z_n$, such that $\iota^*(z_i) = \overline{z}_i$.**

Proof. Step 1: For each basis of 1-forms $\nu_1, ..., \nu_n \in \Lambda^{1,0}_x(M)$, there exists a set of holomorphic coordinate functions $u_1, ..., u_n$ such that $du_i\Big|_x = \nu_i$. To obtain such a coordinate system, **we chose any coordinate system $v_1, ..., v_n$ and apply a linear transform mapping $dv_i\Big|_x$ to ν_i.**

Step 2: The differential $d\iota$ acts on T_xM as a real structure. Using the structure theorem about real structures, we obtain that any real basis $\zeta_1, .., \zeta_n$ of $T^*_x M^\iota$ is a complex basis in the complex vector space $T^*_x M$. Then $\nu_i := \zeta_i + \sqrt{-1}\, I(\zeta_i)\}$ is a basis in $\Lambda^{1,0}_x(M)$. Choose the coordinate system $u_1, ..., u_n$ such that $du_i\Big|_x = \nu_i$ (Step 1). **Replacing u_i by $z_i := u_i + \iota^*(\overline{u}_i)$, we obtain a holomorphic coordinate system z_i on M** (compare with Theorem 1 in Lecture 4) **which satisfies $\iota^*(z_i) = \overline{z}_i$.** ■

DEFINITION: Let $\{U_i\}$ be an complex atlas on M. Assume that any U_i intersecting M^ι satisfies the conclusion of this proposition. Then $\{U_i\}$ is called **compatible with the real structure**.

Real analytic manifolds and real structures

PROPOSITION: Let $M^\iota \subset M$ be a fixed point set of an antiholomorphic involution ι on a complex manifold M, $\{U_i\}$ a complex analytic atlas, and $\Psi_{ij} : U_{ij} \longrightarrow U_{ij}$ the gluing functions. Assume that the atlas U_i is compatible with the real structure, in the sense of the previous proposition. **Then all Ψ_{ij} are real analytic on M^ι, and define a real analytic atlas on the manifold M^ι.**

Proof: All gluing functions from one coordinate system compatible with the real structure to another **commute with ι, acting on coordinate functions as the complex conjugation.** This gives $\Psi_{ij}(\overline{z}_i) = \overline{\Psi_{ij}(z_i)}$. Therefore, Ψ_{ij} preserve M^ι, and are expressed by real-valued functions on M^ι. ∎

Real analytic manifolds and real structures 2

PROPOSITION: Any real analytic manifold can be obtained from this construction.

Proof. Step 1: Let $\{U_i\}$ be a locally finite atlas of a real analytic manifold M, and $\Psi_{ij} : U_{ij} \longrightarrow U_{ij}$ the gluing maps. We realize U_i as an open ball with compact closure in $\operatorname{Re}(\mathbb{C}^n) = \mathbb{R}^n$. By local finiteness, there are only finitely many such Ψ_{ij} for any given U_i. Denote by B_ε an open ball of radius ε in the n-dimensional real space $\operatorname{im}(\mathbb{C}^n)$.

Step 2: Let $\varepsilon > 0$ be a sufficiently small real number such that all Ψ_{ij} can be extended to gluing functions $\tilde{\Psi}_{ij}$ on the open sets $\tilde{U}_i := U_i \times B_\varepsilon \subset \mathbb{C}^n$. **Then $(\tilde{U}_i, \Psi_{ij})$ is an atlas for a complex manifold $M_{\mathbb{C}}$.** Since all Ψ_{ij} are real, they are preserved by the natural involution acting on B_ε as -1 and on U_i as identity. This involution defines a real structure on $M_{\mathbb{C}}$. Clearly, M is the set of its fixed points. ■

Complexification

DEFINITION: Let $M_{\mathbb{R}}$ be a real analytic variety, and $M_{\mathbb{C}}$ a complex analytic variety equipped with an antiholomorphic involution, such that $M_{\mathbb{R}}$ is the set of its fixed points. Then $M_{\mathbb{C}}$ is called **complexification** of $M_{\mathbb{R}}$.

DEFINITION: Let $K \subset M$ be a closed subset of a complex manifold, homeomorphic to $K_1 \subset M_1$, where M_1 is also a complex manifold. Fixing the homeomorphism $K \cong K_1$, we may identify these sets and consider K as a subset M_1. We say that M and M_1 **have the same germ in K** if there exist biholomorphic open subsets $U_1 \subset M_1$ and $U \subset M$ containing K, with the biholomorphism $\varphi: U \longrightarrow U_1$ identity on K.

DEFINITION: **Germ of a variety M in $K \subset M$** is an equivalence class of open subsets $U \subset M$ containing K, with this equivalence relation.

THEOREM: **(Grauert)** Consider category $\mathcal{C}_\iota$, with objects complex varieties (M, ι) equipped with a real structure, and morphisms holomorphic maps commuting with ι. Then **the category of real analytic varieties is equivalent to the category of germs of $M \in \mathcal{C}_\iota$ in $M^\iota \subset M$.**

Adic topology

DEFINITION: Let R be a local ring, and $\mathfrak{m}$ its maximal ideal. **The adic topology** on R is topology with the base of open sets $a+\mathfrak{m}^l$, for all $a \in \mathbb{R}$, $l \in \mathbb{Z}^{>0}$.

REMARK: An **adic completion** is the completion with respect to this topology.

EXERCISE: Suppose that the homomorphism $R \longrightarrow R/\mathfrak{m} = k$ admits a section, $k \hookrightarrow R$. Prove that **the adic completion $\hat{R}$ of R is naturally isomorphic to the completion of $\bigoplus_i \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}$.**

REMARK: We should think about $\hat{R}$ as of the ring of formal power series.

REMARK: Suppose that $M_{\mathbb{R}}$ is a real variety underlying the complex variety M. Then $\mathcal{O}_{M_{\mathbb{R}}}$ is a real part of the completion of $\mathcal{O}_M \otimes_{\mathbb{C}} \overline{\mathcal{O}}_M$. with respect to the uniform topology. The adic topology is stronger than the uniform topology, hence **the adic completion of $\mathcal{O}_{M_{\mathbb{R}}}$ is isomorphic to the completion of $\mathcal{O}_M \otimes_{\mathbb{C}} \overline{\mathcal{O}}_M$**, identified with formal power series of the holomorphic and antiholomorphic variables on M.

Real analytic differentials

DEFINITION: Let B be an open ball in $\mathbb{R}^n$. For an ideal $I \subset \mathcal{O}(B)$ we define **the module of real analytic differentials** on $\mathcal{O}(B)/I$ as $\Omega^1(\mathcal{O}(B)/I) := \Omega^1(\mathcal{O}(B)) \Big/ \Big(I \cdot \Omega^1(\mathcal{O}(B)) + dI \Big)$.

CLAIM: Let X be a complex variety, $X_{\mathbb{R}}$ the underlying real analytic space. **Then the natural sheaf homomorphism $i: \mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X \longrightarrow \mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$ is injective.** For each point $x \in X$, **i induces an isomorphism on x-completions** of $\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X$ and $\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$.

Proof: The adic completions of $\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X$ and $\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$ are equal. ■

COROLLARY: Tensoring both sides of $\Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X) \longrightarrow \Omega^1(\mathcal{O}_{X_{\mathbb{R}}}) \otimes \mathbb{C}$ by $\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}$ **produces an isomorphism**

$$\Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X) \bigotimes_{\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X} \Big(\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C} \Big) = \Omega^1(\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}).$$

■

The complex structure operator on underlying variety

REMARK: This implies that **the sheaf $\Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X)$ admits a canonical decomposition:**

$$\Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X) = \left(\Omega^1(\mathcal{O}_X) \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X\right) \oplus \left(\mathcal{O}_X \otimes_{\mathbb{C}} \Omega^1(\overline{\mathcal{O}}_X\right).$$

DEFINITION: Let $\tilde{I}$ be an endomorphism of $\Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X)$ which acts as a multiplication by $\sqrt{-1}$ on the first component and as a multiplication by $-\sqrt{-1}$ on the second component. We extend $\tilde{I}$ to

$$\Omega^1(\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}) = \Omega^1(\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X) \bigotimes_{\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X} \left(\mathcal{O}_{X_{\mathbb{R}}} \otimes \mathbb{C}\right)$$

using the previous claim. Clearly, $\tilde{I}$ is *real*, that is, comes from the $\mathcal{O}_{X_{\mathbb{R}}}$-linear endomorphism of $\Omega^1(\mathcal{O}_{X_{\mathbb{R}}})$. Denote this $\mathcal{O}_{X_{\mathbb{R}}}$-linear endomorphism by $I: \Omega^1(\mathcal{O}_{X_{\mathbb{R}}}) \longrightarrow \Omega^1(\mathcal{O}_{X_{\mathbb{R}}})$; by construction, $I^2 = -\operatorname{Id}$. The endomorphism I is called **the complex structure operator on the underlying real analytic space**. In the case when X is smooth, I coincides with the usual complex structure operator on the cotangent bundle.

DEFINITION: Let M be a weak real analytic space, and $I: \Omega^1(\mathcal{O}_M) \longrightarrow \Omega^1(\mathcal{O}_M)$ an endomorphism satisfying $I^2 = -1$. Then I is called **an almost complex structure on M**.

Integrability of almost complex structures

THEOREM: X, Y be complex analytic varieties, and $f_{\mathbb{R}}: X_{\mathbb{R}} \longrightarrow Y_{\mathbb{R}}$ be a morphism of underlying real analytic varieties which commutes with the complex structure. **Then there exist a morphism $f: X \longrightarrow Y$ of complex analytic varieties, such that $f_{\mathbb{R}}$ is its underlying morphism.**

Proof: **A continuous map of complex varieties, holomorphic outside of singularities, is meromorphic.** Therefore, its graph $\Gamma \subset X \times Y$ is a complex subvariety of $X \times Y$. Since f is real analytic the projection $\Gamma \xrightarrow{\pi} X$ induces an isomorphism on completions. Therefore, π is flat and unramified, hence etale. Since π is one-to-one on points, π etale implies that π is an isomorphism. ■

DEFINITION: Let M be a real analytic variety, and $I: \Omega^1(\mathcal{O}_M) \longrightarrow \Omega^1(\mathcal{O}_M)$ an endomorphism satisfying $I^2 = -1$. Then I is called **an almost complex structure on M**. If there exist a structure $\mathfrak{C}$ of complex variety on M such that I appears as the complex structure operator associated with $\mathfrak{C}$, we say that I is **integrable**. The above theorem implies that this complex structure is unique if it exists.

QUESTION: **Is there a version of Newlander-Nirenberg theorem in this setup?**

Hyperkähler manifolds

DEFINITION: (E. Calabi, 1978)
Let (M,g) be a Riemannian manifold equipped with three complex structure operators $I,J,K: TM \longrightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}$. Suppose that I, J, K are Kähler. Then (M,I,J,K,g) is called **a hyperkähler manifold**.

LEMMA: Let (M,I,J,K) be hyperkähler. **Then the form $\Omega := \omega_J + \sqrt{-1}\omega_K$ is a holomorphic symplectic 2-form on (M,I).**

Hyperkähler geometry is essentially the same as holomorphic symplectic geometry

THEOREM: (E. Calabi, 1952, S.-T. Yau, 1978)
Let M be a compact, holomorphically symplectic manifold admitting a Kähler metric. **Then M admits a hyperkähler metric, which is uniquely determined by the cohomology class of its Kähler form.**

HYPERCOMPLEX MANIFOLDS

a. k. a "Hyperkähler manifolds without a metric"

DEFINITION: Let M be a smooth manifold equipped with endomorphisms $I, J, K: TM \longrightarrow TM$, satisfying the quaternionic relation $I^2 = J^2 = K^2 = IJK = -\operatorname{Id}$. Suppose that I, J, K are integrable almost complex structures. Then (M, I, J, K) is called **a hypercomplex manifold**.

EXAMPLES:
1. In dimension 1 (real dimension 4), we have **a complete classification of compact hypercomplex manifolds,** due to C. P. Boyer (1988).

2. **Many homogeneous examples,** due to D. Joyce and physicists Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen (1980-ies, early 1990-ies).

3. Some nilmanifolds and solvmanifolds admit **locally homogeneous hypercomplex structure** (M. L. Barberis, I. Dotti, A. Fino) (1990-ies).

4. Some **inhomogeneous examples** are constructed by deformation or as fiber bundles.

In dimension > 1, **no classification results are known** *(and no conjectures either).*

OBATA CONNECTION

Hypercomplex manifolds can be characterized in terms of holonomy

THEOREM: (M. Obata, 1952) Let (M, I, J, K) be a hypercomplex manifold. **Then M admits a unique torsion-free affine connection preserving I, J, K.**

Converse is also true. Suppose that I, J, K are operators defining quaternionic structure on TM, and ∇ a torsion-free, affine connection preserving I, J, K. Then I, J, K are integrable almost complex structures, and (M, I, J, K) is hypercomplex.

Holonomy of Obata connection lies in $GL(n, \mathbb{H})$. Conversely, **a manifold equipped with an affine, torsion-free connection with holonomy in $GL(n, \mathbb{H})$ is hypercomplex**.

This can be used as a definition of a hypercomplex structure.

Twistor spaces

DEFINITION: Induced complex structures on a hypercomplex manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors (Lecture 17).

The **twistor space** $\operatorname{Tw}(M)$ of a hypercomplex manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

DEFINITION: Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$. The operator $\mathcal{I} := I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $\mathcal{I}^2 = -\operatorname{Id}$. **It defines an almost complex structure on the manifold $\operatorname{Tw}(M)$,** which is called **the twistor space** of M. This almost complex structure is integrable.

EXAMPLE: If $M = \mathbb{H}^n$, the space $\operatorname{Tw}(M)$ is biholomorphic to $\operatorname{Tot}(\mathcal{O}(1)^{\oplus 2n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$.

Space of sections

REMARK: Let $Z \subset M$ be a complex subvariety of M. Recall that the set $D(Z)$ of all complex deformations of Z is equipped with a natural structure of a complex manifold, called **the Douady space** of Z in M. If Z is smooth, the Zariski tangent space $T_{[Z]}D(Z)$ is naturally identified with the space of sections of the normal bundle $H^0(NZ)$. Note that **the Douady space is not necessarily reduced**, it might have nilpotents in its structure sheaf.

DEFINITION: Let $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$ be the twistor space of a hypercomplex manifold. The space of holomorphic section of π is called **the space of twistor sections**, or **the space of twistor lines**, denoted by $\operatorname{Sec}(M)$.

DEFINITION: Fix $m \in M$, and consider a twistor section $I \xrightarrow{s_m} (I \times m) \in \mathbb{C}P^1 \times M = \operatorname{Tw}(M)$. Then s_m is called **a horizontal twistor secton.** The variety of horisontal twistor sections is denoted by Hor; it is a real analytic subvariety in the corresponding Douady space.

CLAIM: Let $\iota_0 : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be the anticomplex involution with no fixed points given by the central symmetry of $S^2 \subset \mathbb{R}^3$, and $\iota : \operatorname{Tw} \longrightarrow \operatorname{Tw}$ map (s, m) to $(\iota_0(s), m)$. **Then ι is also an anticomplex involution.** Moreover, the space of horizontal twistor sections **is naturally identified with a connected component the real analytic variety Sec^ι of all twistor sections fixed by ι.**

The twistor data

DEFINITION: Let M be a hypercomplex manifold. The following collection of data is called **the twistor data** associated with M.
* A complex analytic variety Tw, equipped with a morphism π : Tw $\longrightarrow \mathbb{C}P^1$.
* An anticomplex involution ι : Tw $\longrightarrow$ Tw such that $\iota \circ \pi = \pi \circ \iota_0$
* A choice of connected component Hor of $\operatorname{Sec}^\iota \subset \operatorname{Sec}$.

CLAIM: The twistor data of a hypercomplex manifold satisfies the following axioms.

(i) For each point $x \in$ Tw, **there is a unique line** $s \in \operatorname{Hor} \subset \operatorname{Sec}^\iota$, **passing through** x**.**

(ii) For every line $s \in \operatorname{Hor} \subset \operatorname{Sec}^\iota$, **the conormal sheaf** $N^*_s = \ker \left(\Omega^1 \operatorname{Tw}\Big|_{\operatorname{im} s} \xrightarrow{s^*} \Omega^1(\operatorname{im} s)\right)$ **of** $\operatorname{im} s$ **is isomorphic to** $\mathcal{O}(-1)^{\oplus 2n}$**.**

Proof. Step 1: The second condition is implied by $\operatorname{Tw}(\mathbb{H}^n) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus 2n})$. Every hypercomplex manifold can be deformed to $\mathbb{H}^n$ by rescaling (or taking the associative graded quotient), hence **the normal bundle to** $s \in$ Hor **is also** $\mathcal{O}(1)^{\oplus 2n}$**.**

Step 2: The first condition follows from the axiom (ii) of the twistor data and the dimension count. ■

The twistor data of Deligne-Simpson type

THEOREM: (HKLR) **The hypercomplex structure on a hypercomplex manifold M is determined by its twistor data.**

REMARK: The condition (ii) can be replaced by the following condition.

(ii′) For every line $s \in \mathrm{Hor} \subset \mathrm{Sec}^\iota$, and every $a \neq b \in \mathbb{C}P^1$, there exists a tubular neighbourhood $U \subset \mathrm{Tw}$ of $\mathrm{im}\, s$, such that for every $x, y \in U$, $\pi(x) = a, \pi(y) = b$, there exists a unique twistor line $s_{x,y} \subset U$ passing through x and y.

REMARK: The condition (ii) **should be thought of as a linearization of (ii′).** These conditions are equivalent (an exercise).

THEOREM: Let $(\mathrm{Tw}, \pi, \iota, \mathrm{Hor})$ be the twistor data satisfying condition (i). **Then the conditions (ii) and (ii′) are equivalent.**

Proof: Left as a (non-trivial) exercise in deformation theory. ■

DEFINITION: The twistor data satisfying (i), (ii′) define **a twistor space of Deligne-Simpson type**. These conditions were proposed by Deligne and Simpson in order to define **the singular hyperkähler manifolds.**

Hypercomplex varieties

DEFINITION: Let M be a real analytic variety equipped with almost complex structures I, J and K, such that $I\circ J = -J\circ I = K$. Then M is called **an almost hypercomplex variety.**

DEFINITION: An almost hypercomplex variety is equipped with an action of quaternion algebra in its differential sheaf. Each quaternion $L\in \mathbb{H}$, $L^2=-1$ defines an almost complex structure on M. Such an almost complex structure is called **induced by the hypercomplex structure**.

DEFINITION: Let M be an almost hypercomplex variety. We say that M is **hypercomplex** if there exist a pair of induced complex structures $I_1, I_2\in \mathbb{H}$, $I_1\neq \pm I_2$, such that I_1 and I_2 are integrable.

A caution: Not everything which looks hypercomplex satisfies the conditions of this definition. Take a quotient M/G of a hypercomplex manifold by an action of a finite group G, acting compatible with the hyperkähler structure. **Then M/G is not hypercomplex, unless G acts freely.**

EXAMPLE: Recall that a closed subset $Z\subset (M,I,J,K)$ is called **trianalytic** if Z is complex analytic with respect to I,J,K. **Trianalytic subvarieties of hypercomplex manifolds are hypercomplex** (we leave this assertion as an easy exercise in local algebra).

Main results about the hypercomplex varieties

THEOREM: (Desingularization theorem)
Let (M, I, J, K) be a hypercomplex variety, and L an integrable induced complex structure. **Then the normalization $\tilde{M}$ of (M, L) is smooth.** Moreover, $\tilde{M}_L$, as a real analytic variety, is independent from $L \in \mathbb{C}P^1$, and **the complex structures I, J, K induce a hypercomplex structure on $\tilde{M}$,** in such a way that **the normalization map $\tilde{M} \longrightarrow M$ is a morphism of hypercomplex varieties.**

Proof: Later today. ■

THEOREM: Let M be a hypercomplex variety, and L an induced complex structure. **Then L is integrable.** Moreover, its twistor space **is also integrable.**

Proof: Follows from the desingularization. ■

THEOREM: Consider the functor F associating to a hypercomplex variety M the twistor data on its twistor space. **Then F is equivalence of categories:** a hypercomplex variety can be recovered unambiguously from its twistor data.

Proof: Follows from the desingularization and HKLR. ■

Trianalytic subvarieties

DEFINITION: A complex structure $L = aI + bJ + cB$, with $a^2 + b^2 + c^2 = 1$, is called **induced complex structure**, or **induced by the quaternion action**.

DEFINITION: Let (M, I, J, K, g) be a hyperkähler manifold. A complex subvariety $Z \subset (M, I)$ is called **trianalytic** if it is complex analytic with respect to J and K.

REMARK: A trianalytic subvariety $Z \subset (M, I)$ **is complex analytic with respect to any induced complex structure** $L = aI + bJ + cB$.

THEOREM: Let $Z \subset M$ be a trianalytic subvariety of a hyperkähler manifold M, and Z_0 the set of its smooth points. **Then $Z_0 \subset M$ is totally geodesic.**

Proof: The Ricci curvature of M and Z_0 vanishes because they are hyperkähler and hence Einstein. However, the contribution of the second fundamental form to the Ricci curvature of a submanifold is non-negative, and positive when it is non-zero. Therefore, $\text{Ric}(Z_0) = \text{Ric}(M) = 0$ implies that **the second fundamental form of Z_0 vanishes, which is the same as total geodesicity.** ■

The Zariski tangent cone

DEFINITION: Let M be a complex analytic or real analytic variety, and $\mathfrak{m}_x$ an ideal of a point $x \in M$. The **Zariski tangent space** of M in x is $T_z M := \left(\frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}\right)^*$, and **the Zariski tangent cone** $Z_x M$ is the spectrum of the ring $\bigoplus_i \frac{\mathfrak{m}_x^i}{\mathfrak{m}_x^{i+1}}$

REMARK: The Zariski cone **is realized as a closed $\mathbb{C}^*$ or $\mathbb{R}^*$-invariant affine subvariety in $T_x M$.**

REMARK: The Zariski tangent cone might contain nilpotents; **its reduction $\overline{Z}_x M$ is a subvariety in the Zariski tangent space $T_x X$**

CLAIM: Let X be a complex variety, and $X_{\mathbb{R}}$ its underlying real variety. Then $(\overline{Z}_x X)_{\mathbb{R}}$ **is naturally isomorphic to $\overline{Z}_x(X_{\mathbb{R}})$.**

Proof: Locally we can always assume that $X \subset W = \mathbb{C}^n$ is a complex subvariety. Then $T_x X$ is a subspace in W. A vector $l \in T_x X$ belongs to $\overline{Z}_x X \subset T_x X$ if and only if the line $\mathbb{R} \cdot l \subset T_x X$ satisfies $d(tl, X) = o(t)$. ■

The Zariski tangent cone of hypercomplex varieties

COROLLARY: Let M be a hypercomplex variety, I an integrable induced complex structure, and $\overline{Z}_x(M,I) \subset T_x M$ the reduced Zariski tangent cone to (M,I) in $x \in M$. We consider $T_x M$ as a flat hypercomplex manifold. Then **the subvariety $\overline{Z}_x(M,I) \subset T_x M$ is independent from the choice of integrable induced complex structure I.** Moreover, $Z_x(M,I)$ **is a trianalytic subvariety of $T_x M$.**

Proof: Immediately follows from the previous claim. ■

COROLLARY: The reduced Zariski tangent cone $\overline{Z}_x(M,I)$ of a hypercomplex variety **is a union of quaternionic subspaces in $T_x M$**.

Proof: It is trianalytic, hence totally geodesic outside of its singularities. A totally geodesic submanifold in $\mathbb{R}^n$ is an affine subspace. ■

Spaces with locally homogeneous singularities

DEFINITION: Let A be a local ring. Denote by $\mathfrak{m}$ its maximal ideal. Let A_{gr} be the corresponding associated graded ring for the $\mathfrak{m}$-adic filtration. Let $\hat{A}$, $\widehat{A_{gr}}$ be the $\mathfrak{m}$-adic completion of A, A_{gr}. Let $(\hat{A})_{gr}$, $(\widehat{A_{gr}})_{gr}$ be the associated graded rings, which are naturally isomorphic to A_{gr}. We say that A **has locally homogeneous singularities** (LHS) if there exists an isomorphism $\rho: \ \hat{A} \longrightarrow \widehat{A_{gr}}$ which induces the standard isomorphism $i: \ (\hat{A})_{gr} \longrightarrow (\widehat{A_{gr}})_{gr}$ on associated graded rings.

DEFINITION: Let X be a complex or real analytic space. Then X is called **a space with locally homogeneous singularities** (SLHS) if for each $x \in X$, the local ring $\mathcal{O}_x X$ has locally homogeneous singularities.

CLAIM: Let A be a complete local Noetherian ring over $\mathbb{C}$, with a residual field $\mathbb{C}$. Then A **has LHS if and only if there exists a surjective ring homomorphism** $\rho: \ \mathbb{C}[[x_1, ..., x_n]] \longrightarrow A$**, where** $\mathbb{C}[[x_1, ..., x_n]]$ **is the ring of power series, and the ideal** $\ker \rho$ **is homogeneous in** $\mathbb{C}[[x_1, ..., x_n]]$**.**

Proof: Clear. ■

Homogenizing automorphisms

DEFINITION: Let A be a local Noetherian ring over $\mathbb{C}$, with a residual field $\mathbb{C}$, equipped with an automorphism $e: A \longrightarrow A$. Let $\mathfrak{m}$ be a maximal ideal of A. Assume that e acts on $\mathfrak{m}/\mathfrak{m}^2$ as a multiplication by $\lambda \in \mathbb{C}$, $|\lambda| < 1$. Then e is called **a homogenizing automorphism of A**.

PROPOSITION: Let A be a complete Noetherian ring over $\mathbb{C}$, with a residual field $\mathbb{C}$, equipped with a homogenizing authomorphism $e: A \longrightarrow A$. **Then A has locally homogeneous singularities.**

Proof: Take a collection of root vectors $z_1, ..., z_n$ in the maximal ideal $\mathfrak{m}$ such $\underline{z}_i := z_i \mod \mathfrak{m}^2$ generate $\frac{\mathfrak{m}}{\mathfrak{m}^2}$ and are linearly independent. Then A is a completion of its subring generated by $z_1, ..., z_n$, which is graded by the eigenvalues of e. ∎

Homogenizing automorphism on a local ring of a hypercomplex variety

Let M be a hypercomplex variety, and $A_I := \hat{\mathcal{O}}_x(M,I)$ the adic completion of the local ring $\mathcal{O}_x(M,I)$, and $A_{\mathbb R} := \mathcal{O}_x(\widehat{M_{\mathbb R}) \otimes_{\mathbb R}} \mathbb{C}$ be the x-completion of the ring of germs of real analytic complex-valued functions on M.

REMARK: The natural map of completions $U: \ A_I \widehat{\otimes_{\mathbb C}} A_{-I} \longrightarrow A_{\mathbb R}$ **is surjective by Nakayama lemma.**

Denote by p the natural quotient map $p: \ A_{-I} \longrightarrow \mathbb{C}$. By Nakayama again, the projection $A_I \widehat{\otimes_{\mathbb C}} A_{-I} \longrightarrow A_I, \quad a\otimes b \mapsto a \otimes p(b)$, also surjective. It is not hard to see that the kernel of this map contains the kernel of U, which defines a map $e_I: \ A_{\mathbb R} \longrightarrow A_I$. Let $i_I: \ A_I \longrightarrow A_R$ be the map $f \mapsto f\otimes 1$ composed with U.

THEOREM: Let M be a hypercomplex variety, $x\in M$ a point, and I, J induced complex structures, such that $I\neq J$ and $I \neq -J$. Consider the map $\Psi_{I,J}: \ A_I \longrightarrow A_I$ defined as $i_I\circ e_J \circ i_J \circ i_I$. **Then $\Psi_{I,J}$ is a homogenizing automorphism of A_I.** In particular, **a hypercomplex variety is a space with LHS**.

Proof: Left as an exercise. ■

The proof of the desingularization theorem

THEOREM: (Desingularization theorem)
Let (M, I, J, K) be a hypercomplex variety, and L an integrable induced complex structure. **Then the normalization $\tilde{M}$ of (M, L) is smooth.** Moreover, $\tilde{M}_L$, as a real analytic variety, is independent from $L \in \mathbb{C}P^1$, and **the complex structures I, J, K induce a hypercomplex structure on $\tilde{M}$,** in such a way that **the normalization map $\tilde{M} \longrightarrow M$ is a morphism of hypercomplex varieties.**

Proof: As shown above, M is SLHS, hence it is locally isomorphic to its Zariski tangent cone, which is biholomorphic to a union of complex subspaces. Then its normalization is smooth. ■