Lagrangian fibrations and special Kähler geometry

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Differential Geometry seminar,

IMPA, June 11, 2019

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Flat affine manifolds

DEFINITION: A flat affine manifold is a smooth manifold equipped with a flat, torsion-free connection.

EXAMPLE: Let Γ be a group acting on an open subset $U \subset \mathbb{R}^n$ freely, properly discontinuously, by affine transforms. **Then U/Γ is affine.**

REMARK: All flat affine manifolds are obtained this way.

DEFINITION: A flat affine manifold (M, ∇) is called **special affine** if it admits a ∇-invariant volume form, or, equivalently, it its holonomy belongs to $SL(n, \mathbb{R})$. A flat affine manifold is **complete** if it is a quotient of $\mathbb{R}^n$ by a discrete group of affine transforms.

CONJECTURE: Markus conjecture (1961)
a compact affine manifold is complete if and only if it is special.

CONJECTURE: Auslander conjecture (1964)
For any complete affine manifold $\mathbb{R}^n/\Gamma$, the group Γ is solvable.

CONJECTURE: Chern conjecture (1955)
The Euler class of a compact affine manifold vanishes (proven by Bruno Klingler in 2017 for special affine manifolds).

Hessian manifolds

DEFINITION: Let (M,∇) be a flat affine manifold. Since ∇ is torsion-free, the tensor $\nabla^2(f)$ is symmetric for any $f\in C^\infty M$. A Riemannian metric g on M is called **Hessian** if locally $g=\nabla^2(f)$, for some f which is called **the potential** of the metric.

EXERCISE: Prove that **a metric g on (M,∇) is Hessian if and only if $\nabla(g)$ is a symmetric 3-form.**

EXERCISE: Prove that the complex

$$C^\infty(M)\xrightarrow{\nabla^2}\mathrm{Sym}^2 T^*M\xrightarrow{\nabla}\frac{\Lambda^1(M)\otimes\mathrm{Sym}^2 T^*M}{\mathrm{Sym}^3 T^*M}$$

is elliptic.

REMARK: Cohomology of this complex are called **Hessian cohomology**, and the cohomology classes represented by Hessian metrics are called **Hessian classes**.

REMARK: This geometry **is in many ways similar to Kähler geometry.**

Kähler manifolds

DEFINITION: Let (M, I, g) be a complex Hermitian manifold, with $I : TM \longrightarrow TM, I^2 = -\mathrm{Id}$ being the complex structure operator, and g an I-invariant Riemannian form. Then $\omega(\cdot, \cdot) := g(\cdot, I\cdot)$ is an anti-symmetric form, called **fundamental form** of (M, I, g). If it is closed, (M, I, g) is called **Kähler**.

THEOREM: (the local dd^c-lemma)
Let (M, I, g, ω) be a Kähler manifold, and $d^c := IdI^{-1}$. Then locally ω is equal to $dd^c(f)$ for some function f, which is called **a Kähler potential**.

REMARK: Let M be a complex manifold. Then the complex

$$C^\infty(M) \xrightarrow{dd^c} \Lambda^{1,1}(M) \xrightarrow{d} \Lambda^3(M)$$

is elliptic. If, in addition, M is compact and Kähler, its cohomology is equal to $H^{1,1}(M)$. For a general complex manifold, its cohomology is called **Bott-Chern cohomology**.

Calabi-Yau theorem

DEFINITION: A cohomology class of a Kähler form in $H^2(M)$ is called **the Kähler class** of a Kähler manifold.

THEOREM: (Calabi-Yau theorem) Let (M, I) be a compact complex manifold. **A Kähler form ω is uniquely determined by its Kähler class and its volume form.**

This theorem has a Hessian counterpart.

THEOREM: (S.-Y. Cheng, S.-T. Yau, 1982)
Let (M, ∇) be a compact special affine manifold. **A Hessian metric on M is uniquely determined by its Hessian class and its volume form.**

Special Kähler manifolds

In physics, the following structure occurs very often.

DEFINITION: Let $(M, I, \nabla, g, \omega)$ be a Kähler manifold equipped with a flat, torsion-free connection ∇ which satisfies $\nabla(\omega) = 0$ (but does not necessarily preserve g). Assume that g is Hessian. Then $(M, I, \nabla, g, \omega)$ is called **a special Kähler manifold**.

REMARK: A cotangent space to a special Kähler manifold **is always hyperkähler (see the next slide).**

EXAMPLE: The base of an "algebraic Hamiltonion system" **is always equipped with a special Kähler structure** (Donagi, Markman).

EXAMPLE: The moduli space of 3-dimensional Calabi-Yau manifolds **is a special pseudo-Kähler manifold** ("pseudo" here refers to the Kähler metric having Lorentzian signature).

Hyperkähler manifolds

DEFINITION: A **hypercomplex manifold** is a manifold M equipped with three complex structure operators I, J, K, satisfying **quaternionic relations**

$$IJ = -JI = K, \quad I^2 = J^2 = K^2 = -\operatorname{Id}_{TM}$$

(the last equation is a part of the definition of almost complex structures).

DEFINITION: A **hyperkähler manifold** is a hypercomplex manifold equipped with a metric g which is Kähler with respect to I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

Let (M, g, I, J, K) be hyperkähler, and $\omega_I(\cdot,\cdot) := g(\cdot, I\cdot)$ $\omega_J(\cdot,\cdot) := g(\cdot, J\cdot)$, $\omega_K(\cdot,\cdot) := g(\cdot, K\cdot)$ the corresponding Kähler forms.

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: All hyperkähler manifolds are holomorphically symplectic. Indeed, $Sp(n)$ is a compact form of the symplectic group $Sp(2n,\mathbb{C})$, and **the form $\omega_J + \sqrt{-1}\,\omega_K$ is holomorphically symplectic on (M,I).**

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

Holomorphic Lagrangian fibrations

THEOREM: (Matsushita, 1997)
Let $\pi: M \longrightarrow X$ be a surjective holomorphic map from a hyperkähler manifold M to X, whith $0 < \dim X < \dim M$. **Then** $\dim X = 1/2 \dim M$**, and the fibers of** π **are holomorphic Lagrangian** (this means that the symplectic form vanishes on $\pi^{-1}(x)$).

DEFINITION: Such a map is called **holomorphic Lagrangian fibration**.

REMARK: The base of π **is conjectured to be rational.** Hwang (2007) proved that $X \cong \mathbb{C}P^n$, if it is smooth. Matsushita (2000) proved that it has the same rational cohomology as $\mathbb{C}P^n$.

REMARK: The base of π **has a natural flat, torsion-free connection on the smooth locus of** π**.** The combinatorics of this connection can be used to determine the topology of M (Strominger-Yau-Zaslow, Kontsevich-Soibelman).

If we want to learn something about M**, it's recommended to start from a holomorphic Lagrangian fibration.**

Trianalytic subvarieties and hyperholomorphic bundles

Let (M, g, I, J, K) be a hyperkähler manifold. Define **trianalytic subvarieties** as closed subsets which are complex analytic with respect to I, J, K.

0. Trianalytic subvarieties are singular hyperkähler.

1. Let $L \in \mathbb{H}$ be a generic complex structure induced by quaternions. Then all compact complex subvarieties of (M, L) are trianalytic.

2. A normalization of a hyperkähler variety is smooth and hyperkähler. This gives a desingularization ("hyperkähler Hironaka").

3. A complex deformation of a trianalytic subvariety is again trianalytic, the corresponding moduli space is (singularly) hyperkähler.

4. Similar results (also very strong) are true for vector bundles which are holomorphic under I, J, K **("hyperholomorphic bundles")**

Special Lagrangian subvarieties

DEFINITION: (Harvey-Lawson)
Let (M, g, ω, I) be an n-dimensional Calabi-Yau manifold, and $\Omega \in \Lambda^{n,0}(M)$ its holomorphic volume form. **Special Lagrangian subvariety** is a Lagrangian subvariety $S \subset (M, \omega)$ such that $\operatorname{Re}\Omega$ is equal to the Riemannian volume form on S.

THEOREM: (McLean)
Deformations of special Lagrangian subvarieties are unobstructed, and in a neighbourhood of $S \subset M$ this deformation space **is locally identified with $H^1(S, \mathbb{R})$.**

REMARK: This implies that **any special Lagrangian torus belongs to a special Lagrangian fibration, defined in its neighbourhood.**

REMARK: All known examples of special Lagrangian fibrations **are obtained by "hyperkähler rotation":** one takes a holomorphic Lagrangian fibration on (M, I), where (M, I, J, K, g) is hyperkähler, and considers it as a real Lagrangian fibration on (M, J, ω_J).

Mirror symmetry

Kontsevich's homological mirror symmetry. Derived category of coherent sheaves on a Calabi-Yau is equivalent to the Fukaya category on the mirror dual Calabi-Yau, with Hamiltonion isotopy classes of Lagrangian subvarieties as objects, and the Floer cohomology as morphisms.

SYZ (Strominger, Yau, Zaslow) interpretation of mirror symmetry. Each Calabi-Yau manifold is equipped with a special Lagrangian toric fibration, dualizing the tori one obtains the mirror dual Calabi-Yau manifold.

Kapustin-Witten duality

DEFINITION: The Hitchin system is the space of stable Higgs bundles on a compact curve C with semisimple structure group G. It is equipped with a Lagranian fibration with proper fibers and a base $\bigoplus_i H^0(C, K_C^{\otimes j_i})$.

Kapustin-Witten duality (a. k. a "Montonen-Olive/geometric Langlands duality") is the duality between Hitchin systems associated with Langlands dual Lie groups.

1. The corresponding holomorphic Lagrangian fibrations **are expected to be SYZ dual.**

2. In place of the Fukaya category on the symplectic side of Mirror Symmetry, one has **a category of holomorphic Lagrangian subvarieties** (called "BAA", "ABA" and "AAB branes").

3. In place of the derived category of coherent sheaves on the complex side of Mirror Symmetry there is a category which has pairs **(trianalytic subvariety, hyperholomorphic bundle on it) as objects;** these are called "BBB branes".

Kapustin-Witten duality is expected to exchange the BBB branes and the holomorphic Lagrangian subvarieties.

Holomorphic Lagrangian fibrations

DEFINITION: Let M be a holomorphically symplectic manifold, and $\pi : M \longrightarrow B$ a proper surjective holomorphic map with B normal and all fibers holomorphically Lagrangian. Then π is called **a holomorphic Lagrangian fibration**.

CLAIM: (Liouville) A smooth fiber F of a holomorphic Lagrangian fibration $\pi : M \longrightarrow B$ is always a torus.

Proof: For any fibration $\pi : M \longrightarrow B$, any smooth fiber $F = \pi^{-1}(x)$ has trivial normal bundle $NF = \pi^*(T_x B)$. For any function on B, its Hamiltonian gives a section of TF. Choose a collection of holomorphic functions such that their Hamiltonians give a basis in TF. Since these Hamiltonians commute, the corresponding vector fields in TF also commute. This gives a locally free action of an abelian Lie group on F, and therefore F is a quotient of an abelian group by a lattice. ■

DEFINITION: Let $\pi : M \longrightarrow B$ be a proper, regular fibration. The flat connection on the bundle $R^i \pi_*(\mathbb{R}_M)$ on B with the fiber $H^i(\pi^{-1}(x)$ for all $x \in B$ is called **the Gauss-Manin connection**.

Special Kähler geometry on the base of holomorphic Lagrangian fibrations

DEFINITION: Let $\pi : M \longrightarrow B$ be a Lagrangian fibration, and $F := \pi^{-1}(x)$ a regular fiber. Then $\pi^* T_x B = NF = T^* F$. This identifies TB with the bundle $R^1\pi_*(\mathbb{R}_M)$ of first cohomology of fibers. Then TB is equipped with a natural flat connection, called **the Arnold-Liouville connection**.

EXERCISE: Prove that **this connection is torsion-free.**

REMARK: Let $\pi : M \longrightarrow B$ be a holomorphic Lagrangian fibration. A Kähler form ω on M restricted to a smooth fiber F of π defines a cohomology class $[\omega] \in H^2(F)$. Since F is a torus, we can consider $[\omega]$ as a 2-form on $R^1\pi_*(\mathbb{R}_M) = TB$. **This form is clearly parallel under the Gauss-Manin connection, and defines a Kähler structure on B.**

THEOREM: (Donagi-Markman, Freed)
Let $\pi : M \longrightarrow B$ be a holomorphic Lagrangian fibration on a Kähler holomorphic symplectic manifold and $B_0 \subset B$ be the complement to the discriminant locus of B. **Then (B_0, ∇, ω) is a special Kähler manifold.**

Lagrangian subvarieties in Hitchin systems

Note that **the Hitchin systems are equipped with a natiral $\mathbb{C}^*$-action which multiplies the Higgs field by a number.** Hitchin proved the following result related to the Kapustin-Witten duality.

THEOREM: (Hitchin) Let $\pi : M \longrightarrow B$ be the Hitchin system. Then **any $\mathbb{C}^*$-invariant holomorphic Lagrangian submanifold has an open set with the structure of a fibration over a $\mathbb{C}^*$-invariant special Kähler submanifold of B, and each fiber is a disjoint union of translates of an abelian subvariety.**

He asked whether this is true in a more general situation.

THEOREM: (Kamenova-V.) Let $\pi : M \longrightarrow B$ be a holomorphic symplectic Kähler manifold equipped with a holomorphic Lagrangian fibration π with compact fibers, $Z \subset M$ its holomorphic Lagrangian subvariety, $X := \pi(Z)$, and $X_0 \subset B$ the set of all regular values of $\pi : Z \longrightarrow X$. Then **$X_0$ is a special Kähler submanifold of B, and each fiber of $\pi : Z \longrightarrow X$ ober X_0 is a disjoint union of translates of the same torus.**

Lagrangian subspaces in $W \oplus W^*$

Lemma 1: Suppose $V \subset W \oplus W^*$ is a Lagrangian vector subspace in $W \oplus W^*$ with standard symplectic structure, and $\pi : W \oplus W^* \longrightarrow W$ the projection. **Then $\pi(V)^\perp = V \cap W^*$,** where $R^\perp \subset W^*$ denotes the annihilator of a subspace $R \subset W$, and W^* is considered as a subspace in $W \oplus W^*$.

THEOREM: (Kamenova-V.) Let $\pi : M \longrightarrow B$ be a holomorphic symplectic Kähler manifold equipped with a holomorphic Lagrangian fibration π with compact fibers, $Z \subset M$ its holomorphic Lagrangian subvariety, $X := \pi(Z)$, and $X_0 \subset B$ the set of all regular values of $\pi : Z \longrightarrow X$. Then **$X_0$ is a special Kähler submanifold of B, and each fiber of $\pi : Z \longrightarrow X$ ober X_0 is a disjoint union of translates of the same torus.**

Proof. Step 1: Let $Z_x := \pi^{-1}(x) \cap Z$, where $x \in X_0$. The holomorphic symplectic form induces non-degenerate pairing between $T_z^\pi M$ (the fiberwise tangent space) and $T_x B$. Therefore, for all $z \in Z_x$, one has $T_z Z_x = \pi(T_z Z)^\perp \cap T_z^\pi M$, hence **the spaces $T_z Z_x$ are translates for all $z \in Z_x$.** Therefore, **Z_x is the union of tori which are translates of the same torus.**

Step 2: It remains only to show that $X_0 \subset B$ **is flat with respect to the Arnold-Liouville connection.** We may identify the vertical tangent space $T^\pi_z M$ with $H^1(\pi^{-1}(x))$. Then $T_z Z_x$ is identified with the cohomology of a compact torus $H^1(Z_x)$, hence it stays constant under deformations of x. **Then** $\pi(T_z Z) = \pi((T_z Z)^\perp)$ **is constant under the Arnold-Liouville connection.**