# Lagrangian fibrations and special Kähler geometry

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## Flat affine manifolds

**DEFINITION:** **A flat affine manifold** is a smooth manifold equipped with a flat, torsion-free connection.

**EXAMPLE:** Let $\Gamma$ be a group acting on an open subset $U \subset \mathbb{R}^n$ freely, properly discontinuously, by affine transforms. **Then $U/\Gamma$ is affine.**

**REMARK:** **All flat affine manifolds are obtained this way.**

**DEFINITION:** A flat affine manifold $(M, \nabla)$ is called **special affine** if it admits a $\nabla$-invariant volume form, or, equivalently, it its holonomy belongs to $SL(n, \mathbb{R})$. A flat affine manifold is **complete** if it is a quotient of $\mathbb{R}^n$ by a discrete group of affine transforms.

**CONJECTURE:** **Markus conjecture** (1961)
a compact affine manifold is complete if and only if it is special.

**CONJECTURE:** **Auslander conjecture** (1964)
For any complete affine manifold $\mathbb{R}^n/\Gamma$, the group $\Gamma$ is solvable.

**CONJECTURE:** **Chern conjecture** (1955)
The Euler class of a compact affine manifold vanishes (proven by Bruno Klingler in 2017 for special affine manifolds).

## Hessian manifolds

**DEFINITION:** Let $(M, \nabla)$ be a flat affine manifold. Since $\nabla$ is torsion-free, the tensor $\nabla^2(f)$ is symmetric for any $f \in C^\infty M$. A Riemannian metric $g$ on $M$ is called **Hessian** if locally $g = \nabla^2(f)$, for some $f$ which is called **the potential** of the metric.

**EXERCISE:** Prove that **a metric $g$ on $(M, \nabla)$ is Hessian if and only if $\nabla(g)$ is a symmetric 3-form.**

**EXERCISE:** Prove that the complex

$$C^\infty(M) \xrightarrow{\nabla^2} \mathrm{Sym}^2 T^*M \xrightarrow{\nabla} \frac{\Lambda^1(M) \otimes \mathrm{Sym}^2 T^*M}{\mathrm{Sym}^3 T^*M}$$

is elliptic.

**REMARK:** Cohomology of this complex are called **Hessian cohomology**, and the cohomology classes represented by Hessian metrics are called **Hessian classes**.

**REMARK:** This geometry **is in many ways similar to Kähler geometry.**

## Special Kähler manifolds

**DEFINITION:** Let $(M, I, g)$ be a complex Hermitian manifold, with $I : TM \longrightarrow TM, I^2 = -\operatorname{Id}$ being the complex structure operator, and $g$ an $I$-invariant Riemannian form. Then $\omega(\cdot,\cdot) := g(\cdot, I\cdot)$ is an anti-symmetric form, called **fundamental form** of $(M, I, g)$. If it is closed, $(M, I, g)$ is called **Kähler**.

In physics, the following structure occurs very often.

**DEFINITION:** Let $(M, I, \nabla, g, \omega)$ be a Kähler manifold equipped with a flat, torsion-free connection $\nabla$ which satisfies $\nabla(\omega) = 0$ (but does not necessarily preserve $g$). Assume that $g$ is Hessian. Then $(M, I, \nabla, g, \omega)$ is called **a special Kähler manifold**.

**REMARK:** A cotangent space to a special Kähler manifold **is always hyperkähler (see the next slide).**

**EXAMPLE:** The base of an "algebraic Hamiltonion system" **is always equipped with a special Kähler structure** (Donagi, Markman).

**EXAMPLE:** The moduli space of 3-dimensional Calabi-Yau manifolds **is a special pseudo-Kähler manifold** ("pseudo" here refers to the Kähler metric having Lorentzian signature).

**REMARK:** Special Kähler manifolds satisfy $\det g = \omega^n$, hence $\nabla \det g = const$. **This implies that they are flat affine Calabi-Yau**, in the sense of Cheng-Yau.

## Hyperkähler manifolds

**DEFINITION:** A **hypercomplex manifold** is a manifold $M$ equipped with three complex structure operators $I, J, K$, satisfying **quaternionic relations**

$$IJ = -JI = K, \quad I^2 = J^2 = K^2 = -\operatorname{Id}_{TM}$$

(the last equation is a part of the definition of almost complex structures).

**DEFINITION:** A **hyperkähler manifold** is a hypercomplex manifold equipped with a metric $g$ which is Kähler with respect to $I, J, K$.

**REMARK:** This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves $I, J, K$.

**DEFINITION:** Let $M$ be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of $M$.

**REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$** (the group of all endomorphisms preserving $I, J, K$).

## Holomorphically symplectic manifolds

Let $(M, g, I, J, K)$ be hyperkähler, and $\omega_I(\cdot,\cdot) := g(\cdot, I\cdot)$ $\omega_J(\cdot,\cdot) := g(\cdot, J\cdot)$, $\omega_K(\cdot,\cdot) := g(\cdot, K\cdot)$ the corresponding Kähler forms.

**DEFINITION:** A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

**REMARK: All hyperkähler manifolds are holomorphically symplectic.** Indeed, $Sp(n)$ is a compact form of the symplectic group $Sp(2n,\mathbb{C})$, and **the form $\omega_J + \sqrt{-1}\,\omega_K$ is holomorphically symplectic on $(M, I)$.**

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

## Holomorphic Lagrangian fibrations

**THEOREM:** (Matsushita, 1997)
Let $\pi: M \longrightarrow X$ be a surjective holomorphic map from a hyperkähler manifold $M$ to $X$, whith $0 < \dim X < \dim M$. **Then $\dim X = 1/2 \dim M$, and the fibers of $\pi$ are holomorphic Lagrangian** (this means that the symplectic form vanishes on $\pi^{-1}(x)$).

**DEFINITION:** Such a map is called **holomorphic Lagrangian fibration**.

**REMARK:** The base of $\pi$ **is conjectured to be rational.** Hwang (2007) proved that $X \cong \mathbb{C}P^n$, if it is smooth. Matsushita (2000) proved that it has the same rational cohomology as $\mathbb{C}P^n$.

**REMARK:** The base of $\pi$ **has a natural flat, torsion-free connection on the smooth locus of $\pi$.** The combinatorics of this connection can be used to determine the topology of $M$ (Strominger-Yau-Zaslow, Kontsevich-Soibelman).

**If we want to learn something about $M$, it's recommended to start from a holomorphic Lagrangian fibration.**

## Trianalytic subvarieties and hyperholomorphic bundles

Let $(M, g, I, J, K)$ be a hyperkähler manifold. Define **trianalytic subvarieties** as closed subsets which are complex analytic with respect to $I$, $J$, $K$.

0. Trianalytic subvarieties are singular hyperkähler.

1. Let $L \in \mathbb{H}$ be a generic complex structure induced by quaternions. Then all compact complex subvarieties of $(M, L)$ are trianalytic.

2. A normalization of a hyperkähler variety is smooth and hyperkähler. This gives a desingularization ("hyperkähler Hironaka").

3. A complex deformation of a trianalytic subvariety is again trianalytic, the corresponding moduli space is (singularly) hyperkähler.

4. Similar results (also very strong) are true for vector bundles which are holomorphic under $I$, $J$, $K$ **("hyperholomorphic bundles")**

## Special Lagrangian subvarieties

**DEFINITION: (Harvey-Lawson)**
Let $(M, g, \omega, I)$ be an $n$-dimensional Calabi-Yau manifold, and $\Omega \in \Lambda^{n,0}(M)$ its holomorphic volume form. **Special Lagrangian subvariety** is a Lagrangian subvariety $S \subset (M, \omega)$ such that $\operatorname{Re}\Omega$ is equal to the Riemannian volume form on $S$.

**THEOREM: (McLean)**
**Deformations of special Lagrangian subvarieties are unobstructed,** and in a neighbourhood of $S \subset M$ this deformation space **is locally identified with $H^1(S, \mathbb{R})$.**

**REMARK:** This implies that **any special Lagrangian torus belongs to a special Lagrangian fibration, defined in its neighbourhood.**

**REMARK:** All known examples of special Lagrangian fibrations **are obtained by "hyperkähler rotation":** one takes a holomorphic Lagrangian fibration on $(M, I)$, where $(M, I, J, K, g)$ is hyperkähler, and considers it as a real Lagrangian fibration on $(M, J, \omega_J)$.

## Mirror symmetry

**Kontsevich's homological mirror symmetry.** Derived category of coherent sheaves on a Calabi-Yau is equivalent to the Fukaya category on the mirror dual Calabi-Yau, with Hamiltonion isotopy classes of Lagrangian subvarieties as objects, and the Floer cohomology as morphisms.

**SYZ (Strominger, Yau, Zaslow) interpretation of mirror symmetry.** Each Calabi-Yau manifold is equipped with a special Lagrangian toric fibration, dualizing the tori one obtains the mirror dual Calabi-Yau manifold.

## Chern connections and holomorphic structure operators

**DEFINITION:** let $(B, \nabla)$ be a smooth bundle with connection and a holomorphic structure $\bar\partial\ B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1}:\ B \longrightarrow \Lambda^{0,1}(M) \otimes B, \quad \nabla^{1,0}:\ B \longrightarrow \Lambda^{1,0}(M) \otimes B.$$

We say that $\nabla$ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \bar\partial$.

**DEFINITION: A Chern connection** on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric. **THEOREM:** On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

**REMARK:** The curvature of a Chern connection on $B$ is an End($B$)-valued (1,1)-form: $\Theta_B \in \Lambda^{1,1}(\mathrm{End}(B))$.

**REMARK: A converse is true**, too. Given a Hermitian connection $\nabla$ on a vector bundle $B$ with curvature in $\Lambda^{1,1}(\mathrm{End}(B))$, we obtain a holomorphic structure operator $\bar\partial = \nabla^{0,1}$. Then, **$\nabla$ is a Chern connection of $(B, \bar\partial)$.**

## Kobayashi-Hitchin correspondence

**DEFINITION:** Let $F$ be a coherent sheaf over an $n$-dimensional compact Kähler manifold $M$. Let

$$\operatorname{slope}(F) := \frac{1}{\operatorname{rank}(F)} \int_M \frac{c_1(F) \wedge \omega^{n-1}}{vol(M)}.$$

A torsion-free sheaf $F$ is called **(Mumford-Takemoto) stable** if for all subsheaves $F' \subset F$ one has $\operatorname{slope}(F') < \operatorname{slope}(F)$. If $F$ is a direct sum of stable sheaves of the same slope, $F$ is called **polystable**.

**DEFINITION:** A Hermitian metric on a holomorphic vector bundle $B$ is called **Yang-Mills** (Hermitian-Einstein) if the curvature of its Chern connection satisfies $\Theta_B \wedge \omega^{n-1} = \operatorname{slope}(F) \cdot \operatorname{Id}_B \cdot \omega^n$, or, equivalently, $\Lambda(\Theta_B) = \operatorname{slope}(F) \cdot \operatorname{Id}_B$. A Yang-Mills connection is a Chern connection induced by the Yang-Mills metric.

**REMARK: Yang-Mills connections minimize the integral**

$$\int_M |\Theta_B|^2 \operatorname{Vol}_M$$

**Kobayashi-Hitchin correspondence:** (Donaldson, Uhlenbeck-Yau). Let $B$ be a holomorphic vector bundle. **Then $B$ admits Yang-Mills connection if and only if $B$ is polystable.** Moreover such a connection is **unique**.

## Higgs bundles

**DEFINITION:** Let $M$ be a complex manifold, and $B$ a holomorphic vector bundle. **Higgs field** on $B$ is and $\operatorname{End}(B)$-valued holomorphic 1-form $\theta \in H^0(\Omega^1(M) \otimes \operatorname{End}(B))$ which satisfies $\theta \wedge \theta = 0$, where $\theta \wedge \theta$ is exterior square, considered as a section of $\Omega^2(M) \otimes \operatorname{End}(B)$; if $\theta = \sum_i v_i \otimes A_i$, then $\theta \wedge \theta = \sum_{i,j} v_i \wedge v_j \otimes [A_i, A_j]$.

**DEFINITION:** Let $(B, \theta)$ be a Higgs bundle. **Higgs subsheaf** is a coherent subsheaf $F \subset B$ such that for each $f \in F$ the section $\theta(f) \in B$ belongs to $F$.

**DEFINITION:** A Higgs bundle $B$ is called **stable** if for all Higgs subsheaves $F \subset B$ one has $\operatorname{slope}(F) < \operatorname{slope}(B)$. If $B$ is a direct sum of stable bundles of the same slope, $B$ is called **polystable**.

## Corlette-Simpson correspondence

**DEFINITION:** Let $\nabla$ be a Chern connection on a Higgs bundle. Its **Higgs connection** is $\nabla_\theta := \nabla + \theta + \overline{\theta}^*$, where $\overline{\theta}^* = \sum_i \overline{v}_i \otimes A_i^T$ for $\theta = \sum_i v_i \otimes A_i$. Its **Higgs curvature** is $\Theta := (\nabla_\theta)^2 \in \Lambda^2(M) \otimes \operatorname{End}(B)$. We say that the Higgs bundle $(B, \nabla, \theta)$ is **Hermitian Yang-Mills**, if $\Lambda\Theta = const\, \operatorname{Id}_B$.

**THEOREM: Corlette-Simpson correspondence:** Let $(B, \theta)$ be a Higgs bundle. **Then $(B, \theta)$ admits Yang-Mills Higgs connection if and only if $(B, \theta)$ is polystable.** Moreover such a connection is **unique**.

**THEOREM: ("Bogomolov-Lübke inequality", due to Corlette and Simpson)** Let $(B, \nabla, \theta)$ be a Yang-Mills Higgs bundle with $c_1(B) = 0, c_2(B) = 0$. **Then the connection $\nabla_\theta := \nabla + \theta + \overline{\theta}^*$ is flat.**

**THEOREM: (Simpson)** Let $M$ be a compact Kähler manifold, and $\pi_1(M) \xrightarrow{\rho} GL(n, \mathbb{C})$ a representation. Assume that the algebraic closure of $\rho(\pi_1(M))$ is a reductive Lie group. **Then $\rho$ is obtained from a polystable Higgs bundle $(B, \theta)$ using the Corlette-Simpson correspondence.** Moreover, **$(B, \theta)$ is determined uniquely from $\rho$.**

## Narasimhan-Seshadri

**THEOREM: (Kobayashi-Lubke)**
The **Yang-Mills connection on a holomorphic bundle with $c_1(B) = 0, c_2(B) = 0$ is flat.**

**COROLLARY: (Narasimhan, Seshadri)**
A stable bundle with $c_1(B) = 0, c_2(B) = 0$ **admits a unique flat unitary Chern connection**

**REMARK:** Let $U \subset \mathcal{X}_D$ be the space of Higgs bundles $(B, \theta)$, where $B$ is stable. Then $U$ is clearly open. Let $Z$ is the space of stable bundles with $c_1(B) = 0, c_2(B) = 0$. **The natural forgetful map $(B, \theta) \longrightarrow B$ is a Lagrangian fibration, which is locally identifies $U$ with $T^*Z$.**

## Non-Abelian Hodge theory

**REMARK:** Let $\gamma_1, ..., \gamma_n \in \pi_1(M)$ be generators of $\pi_1(M)$, and $R_1, ..., R_k$ the relations. The space $\mathrm{Hom}(\pi_1(M), GL(n,\mathbb{C}))$ of complex-valued representations $\pi_1(M) \xrightarrow{\rho} GL(n,\mathbb{C})$ is a set of matrices $A_1, ..., A_n$ such that $R_1(A_1, ..., A_n) = ... = R_m(A_1, ..., A_n) = \mathrm{Id}$. Therefore, $\mathrm{Hom}(\pi_1(M), GL(n,\mathbb{C}))$ is an affine variety. **The moduli of representations of $\pi_1(M)$** is the quotient $\mathrm{Hom}(\pi_1(M), GL(n,\mathbb{C}))/\!/GL(n,\mathbb{C})$ by the adjoint action, where $*/\!/GL(n,\mathbb{C})$ **signifies the GIT reduction** (that is, **the symplectic reduction)**.

**DEFINITION:** Let $\tilde{\mathcal{X}}$ be the subset of $\mathrm{Hom}(\pi_1(M), GL(n,\mathbb{C}))/\!/GL(n,\mathbb{C})$ consisting of representations $\rho$ with reductive Zariski closure of $\mathrm{im}\,\rho$, and $\mathcal{X}_B := \tilde{\mathcal{X}}/\!/GL(n,\mathbb{C})$. Then $\mathcal{X}_B$ is called **the Betti moduli space** of representations of $\pi_1(M)$. The complex structure on $\mathcal{X}_B$ is induced from the complex structure on $\mathrm{Hom}(\pi_1(M), GL(n,\mathbb{C}))$ as indicated above. **The Dolbeault moduli space** $\mathcal{X}_D$ is the same space with different complex structure. The complex structure on the Dolbeault moduli space **is induced from the moduli of Higgs bundles.**

**THEOREM:** **(Simpson)** The moduli space $\mathcal{X}$ **is equipped with a natural hyperkähler structure,** and **the Dolbeault and Betti complex structures are induced from the quaternion action** associated with this hyperkähler structure.

## Kapustin-Witten duality

**DEFINITION: The Hitchin system** is the space of stable Higgs bundles on a compact curve $C$. It is equipped with a Lagranian fibration with proper fibers and a base $\bigoplus_i H^0(C, K_C^{\otimes j_i})$. In a similar way one defines a principal Higgs bundle (which is a holomorphic $G$-bundle $P$ with a Higgs field $\theta \in H^0(\Omega^1(M) \otimes \operatorname{Lie}(P))$ and a Hitchin system on a $G$-bundle over $C$.

**Kapustin-Witten duality (a. k. a "Montonen-Olive/geometric Langlands duality")** is the duality between Hitchin systems associated with Langlands dual Lie groups.

**1. The corresponding** holomorphic Lagrangian fibrations **are expected to be SYZ dual.**
**2. In place of** the Fukaya category on the symplectic side of Mirror Symmetry, one has **a category of holomorphic Lagrangian subvarieties** (called "BAA", "ABA" and "AAB branes").
**3. In place of** the derived category of coherent sheaves on the complex side of Mirror Symmetry there is a category which has pairs **(trianalytic subvariety, hyperholomorphic bundle on it) as objects;** these are called "BBB branes".

**Kapustin-Witten duality is expected to exchange the BBB branes and the holomorphic Lagrangian subvarieties**.

## Holomorphic Lagrangian fibrations

**DEFINITION:** Let $M$ be a holomorphically symplectic manifold, and $\pi : M \longrightarrow B$ a proper surjective holomorphic map with $B$ normal and all fibers holomorphically Lagrangian. Then $\pi$ is called **a holomorphic Lagrangian fibration**.

**CLAIM: (Liouville) A smooth fiber $F$ of a holomorphic Lagrangian fibration $\pi : M \longrightarrow B$ is always a torus.**

**Proof:** For any fibration $\pi : M \longrightarrow B$, any smooth fiber $F = \pi^{-1}(x)$ has trivial normal bundle $NF = \pi^*(T_x B)$. For any function on $B$, its Hamiltonian gives a section of $TF$. Choose a collection of holomorphic functions such that their Hamiltonians give a basis in $TF$. Since these Hamiltonians commute, the corresponding vector fields in $TF$ also commute. This gives a locally free action of an abelian Lie group on $F$, and therefore $F$ is a quotient of an abelian group by a lattice. ■

**DEFINITION:** Let $\pi : M \longrightarrow B$ be a proper, regular fibration. The flat connection on the bundle $R^i\pi_*(\mathbb{R}_M)$ on $B$ with the fiber $H^i(\pi^{-1}(x)$ for all $x \in B$ is called **the Gauss-Manin connection**.

## Special Kähler geometry and algebraic Hamiltonian systems

**DEFINITION:** Let $\pi : M \longrightarrow B$ be a Lagrangian fibration, and $F := \pi^{-1}(x)$ a regular fiber. Then $\pi^* T_x B = NF = T^* F$. This identifies $TB$ with the bundle $R^1\pi_*(\mathbb{R}_M)$ of first cohomology of fibers. Then $TB$ is equipped with a natural flat connection, called **the Arnold-Liouville connection**.

**EXERCISE:** Prove that **this connection is torsion-free.**

**REMARK:** Let $\pi : M \longrightarrow B$ be a holomorphic Lagrangian fibration. A Kähler form $\omega$ on $M$ restricted to a smooth fiber $F$ of $\pi$ defines a cohomology class $[\omega] \in H^2(F)$. Since $F$ is a torus, we can consider $[\omega]$ as a 2-form on $R^1\pi_*(\mathbb{R}_M) = TB$. **This form is clearly parallel under the Gauss-Manin connection, and defines a Kähler structure on $B$.**

**THEOREM: (Donagi-Markman, Freed)**
Let $\pi : M \longrightarrow B$ be a holomorphic Lagrangian fibration on a Kähler holomorphic symplectic manifold (Donagi-Markman call this **"an algebraic Hamiltonian system"**) and $B_0 \subset B$ be the complement to the discriminant locus of $B$. **Then $(B_0, \nabla, \omega)$ is a special Kähler manifold.**

## Lagrangian subvarieties in Hitchin systems

Note that **the Hitchin systems are equipped with a natiral $\mathbb{C}^*$-action which multiplies the Higgs field by a number.** Hitchin proved the following result related to the Kapustin-Witten duality.

**THEOREM:** (Hitchin) Let $\pi\ : M \longrightarrow B$ be the Hitchin system. Then **any $\mathbb{C}^*$-invariant holomorphic Lagrangian submanifold has an open set with the structure of a fibration over a $\mathbb{C}^*$-invariant special Kähler submanifold of $B$, and each fiber is a disjoint union of translates of an abelian subvariety.**

He asked whether this is true in a more general situation.

**THEOREM: (Kamenova-V.)** Let $\pi :\ M \longrightarrow B$ be a holomorphic symplectic Kähler manifold equipped with a holomorphic Lagrangian fibration $\pi$ with compact fibers, $Z \subset M$ its holomorphic Lagrangian subvariety, $X := \pi(Z)$, and $X_0 \subset B$ the set of all regular values of $\pi :\ Z \longrightarrow X$. Then **$X_0$ is a special Kähler submanifold of $B$, and each fiber of $\pi :\ Z \longrightarrow X$ ober $X_0$ is a disjoint union of translates of the same torus.**

## Lagrangian subspaces in $W \oplus W^*$

**Lemma 1:** Suppose $V \subset W \oplus W^*$ is a Lagrangian vector subspace in $W \oplus W^*$ with standard symplectic structure, and $\pi: W \oplus W^* \longrightarrow W$ the projection. **Then $\pi(V)^\perp = V \cap W^*$,** where $R^\perp \subset W^*$ denotes the annihilator of a subspace $R \subset W$, and $W^*$ is considered as a subspace in $W \oplus W^*$.

**THEOREM: (Kamenova-V.)** Let $\pi: M \longrightarrow B$ be a holomorphic symplectic Kähler manifold equipped with a holomorphic Lagrangian fibration $\pi$ with compact fibers, $Z \subset M$ an irreducible, closed holomorphic Lagrangian subvariety, $X := \pi(Z)$, and $X_0 \subset B$ the set of all regular values of $\pi: Z \longrightarrow X$. Then **$X_0$ is a special Kähler submanifold of $B$, and each fiber of $\pi: Z \longrightarrow X$ over $X_0$ is a disjoint union of translates of the same torus.**

**Proof. Step 1:** Let $Z_x := \pi^{-1}(x) \cap Z$, where $x \in X_0$. The holomorphic symplectic form induces non-degenerate pairing between $T_z^\pi M$ (the fiberwise tangent space) and $T_x B$. By Lemma 1, for all $z \in Z_x$, one has $T_z Z_x = \pi(T_z Z)^\perp \cap T_z^\pi M$, hence **the spaces $T_z Z_x$ are translates for all $z \in Z_x$.** Therefore, **$Z_x$ is a disconnected union of subtori of $\pi^{-1}(x)$, and all of these subtori are translates.**

**Step 2:** It remains only to show that $X_0 \subset B$ **is flat with respect to the Arnold-Liouville connection.** We may identify the vertical tangent space $T_z^\pi M$ with $H^1(\pi^{-1}(x))$. Then $T_z Z_x$ is identified with the cohomology of a compact torus $H^1(Z_x)$, hence it stays constant under deformations of $x$. **Then** $\pi(T_z Z) = \pi((T_z Z)^\perp)$ **is constant under the Arnold-Liouville connection.**

*Friedrich Hirzebruch, Nigel Hitchin and Edward Witten, 2009, MFO*