Stable bundles on Vaisman manifolds

Misha Verbitsky

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Alfréd Rényi Institute of Mathematics
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LCK manifolds

DEFINITION: A complex Hermitian manifold (M, I, g, ω) is called **locally conformally Kähler** (LCK) if there exists a closed 1-form θ such that $d\omega = \theta \wedge \omega$. The 1-form θ is called the **Lee form** and the g-dual vector field $\theta^\sharp$ is called the **Lee field**.

REMARK: This definition **is equivalent with the existence of a Kähler cover $(\tilde{M}, \tilde{\omega}) \longrightarrow M$ such that the deck group Γ acts on $(M, \tilde{\omega})$ by holomorphic homotheties.** Indeed, suppose that θ is exact, $df = \theta$. **Then $e^{-f}\omega$ is a Kähler form.**

THEOREM: (Vaisman)
A compact LCK manifold with non-exact Lee form **does not admit a Kähler structure.**

REMARK: Such manifold are called **strict LCK**. Further on, **we shall consider only strict LCK manifolds.**

Vaisman manifolds

DEFINITION: The LCK manifold (M, I, g, ω) is a **Vaisman manifold** if the Lee form is parallel with respect to the Levi-Civita connection.

THEOREM: A compact (strictly) LCK manifold M is Vaisman **if and only if it admits a non-trivial action of a complex Lie group of positive dimension,** acting by holomorphic isometries.

DEFINITION: A linear Hopf manifold is a quotient $M := \frac{\mathbb{C}^n \backslash 0}{\langle A \rangle}$ where A is a linear contraction. When A is diagonalizable, M is called **diagonal Hopf.**

CLAIM: All diagonal Hopf manifolds are Vaisman, and **all non-diagonal Hopf manifolds are LCK and not Vaisman**.

THEOREM: A compact complex manifold admits a Vaisman structure **if and only if it admits a holomorphic embedding to a diagonal Hopf manifold.**

Vaisman manifolds and algebraic cones

DEFINITION: Let L be an ample bundle on a projective orbifold, and $\operatorname{Tot}^\circ(L)$ the set of non-zero vectors in its total space. Then $\operatorname{Tot}^\circ(L)$ is called **an open algebraic cone.**

PROPOSITION: "Calabi formula (2.6)"
Let h be a metric with positive curvature on $\operatorname{Tot}^\circ(L)$. Consider h as a function on $\operatorname{Tot}^\circ(L)$, with $h(v) := (v, v)_h$. **Then $dd^c h$ is a Kähler form on $\operatorname{Tot}^\circ(L)$.**

THEOREM: Let X be a projective orbifold, L an ample bundle, h a metric with positive curvature ω on L (in this case, ω is a Kähler form on X). Consider an isometry $\varphi : X \to X$ which can be extended to a holomorphic isometry of $\operatorname{Tot}^\circ(L)$ (such an isometry is called **linearizable**, or **hamiltonian**), and let γ be the $\mathbb{Z}$-action on $\operatorname{Tot}^\circ(L)$ which takes a vector $v \in \operatorname{Tot}^\circ(L)$ to $\lambda\varphi(v)$, where $\lambda \in \mathbb{C}$, $|\lambda| > 1$. **Then $\frac{\operatorname{Tot}^\circ(L)}{\langle\gamma\rangle}$ is Vaisman, and all Vaisman manifolds are obtained this way.**

Canonical foliation

REMARK: On a Vaisman manifold, the Lee vector field $\theta^\sharp$ and the anti-Lee field $I\theta^\sharp$ **are Killing and holomorphic.** Moreover, they commute: $[\theta^\sharp, I\theta^\sharp] = 0$. Therefore, **they define a holomorphic 1-dimensional foliation Σ.** This foliation is called **the canonical foliation.**

CLAIM: Let (M, ω, θ) be a Vaisman manifold, and $\theta^c := I(\theta($. Then the form $\omega_0 = -d\theta^c$ satisfies $\omega_0 = \omega - \theta \wedge \theta^c$. Moreover, this form is pseudo-Hermitian, invariant with respect to the action generated by $\theta^\sharp, I\theta^\sharp$, **vanishes on Σ and is positive in the transversal direction.**

REMARK: Using the form ω_0, it is easy to see that the canonical foliation Σ **is independent from the choice of the metric.**

REMARK: This form **is transversally Kähler** with respect to Σ.

EXAMPLE: Let $H = \frac{\mathbb{C}^n \backslash 0}{\langle \lambda \rangle}$ be a classical Hopf manifold, and $\pi: H \to \mathbb{C}P^{n-1}$ the standard projection. **Then its fibers are the leaves of the canonical foliation.**
EXAMPLE: Consider a Vaisman manifolds obtained as above, $\frac{\operatorname{Tot}^\circ(L)}{\langle \gamma \rangle}$ such that $\gamma(v) = \lambda\varphi(v)$. **Then its Lee field is tangent to the fibers of the bundle L,** and the corresponding diffeomorphism flow takes v to $e^t v$, for $t \in \mathbb{C}$.

Quasi-regular foliations

DEFINITION: A foliation is called **quasi-regular** if all its leaves are compact. If this is not so, it is called **irregular**. A foliation is called **regular** if all its leaves are compact, and the leaf space is smooth.

DEFINITION: A Vaisman manifold is **regular/quasi-regular/irregular** if its canonical foliation is regular/quasi-regular/irregular.

THEOREM: The quasi-regular Vaisman manifolds **are elliptically fibered over a projective orbifold.**

REMARK: A Vaisman manifolds $\frac{\operatorname{Tot}^\circ(L)}{\langle\gamma\rangle}$ such that $\gamma(v) = \lambda\varphi(v)$ is quasi-regular **if and only if φ is a map of finite order.**

Gauduchon metrics

DEFINITION: A Hermitian metric ω on a complex n-manifold is called **Gauduchon** if $dd^c\omega^{n-1} = 0$.

THEOREM: (P. Gauduchon, 1978) Let M be a compact, complex manifold, and h a Hermitian form. **Then there exists a Gauduchon metric conformally equivalent** to h, and it is unique, up to a constant multiplier.

EXAMPLE: A Vaisman metric is Gauduchon. It follows from an elementary calculation: $*dd^c\omega^{n-1} = const \cdot d^*\theta$.

REMARK: If ω is Gauduchon, then (by Stokes' theorem) $\int_M \omega^{n-1} dd^c f = 0$ for any f. The curvature Θ_L of a holomorphic line bundle L is well-defined up to $dd^c \log |h|$, where h is a conformal factor. Therefore, **for any line bundle L, the number $\deg_\omega L := \int_M \omega^{n-1} \wedge \Theta_L$ is well defined.**

REMARK: Unlike the Kähler case, $\deg_\omega L$ is a holomorphic invariant of L, and **not topological.**

DEFINITION: Given a torsion-free coherent sheaf F of rank r, let $F^* := \mathcal{H}om_{\mathcal{O}_M}(F, \mathcal{O}_M)$ be the dual coherent sheaf. We define **its determinant** as $\det F := \Lambda^r_{\mathcal{O}_M} F^{**}$. From algebraic geometry it is known that $\det F$ is a line bundle. Define **the degree** $\deg_\omega F := \deg_\omega \det F = \int_M \operatorname{Tr} \Theta_F \wedge \omega^{n-1}$.

Kobayashi-Hitchin correspondence

DEFINITION: Let F be a coherent sheaf over an n-dimensional Gauduchon manifold (M,ω), and $\operatorname{slope}(F) := \frac{\deg_\omega F}{\operatorname{rank}(F)}$. A torsion-free sheaf F is called **stable** if for all subsheaves $F' \subset F$ one has $\operatorname{slope}(F') < \operatorname{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

DEFINITION: A Hermitian metric on a holomorphic vector bundle B is called **Yang-Mills** (Hermitian-Einstein) if $\Theta_B \wedge \omega^{n-1} = \operatorname{slope}(F) \cdot \operatorname{Id}_B \cdot \omega^n$, where Θ_B is its curvature.

THEOREM: (**Kobayashi-Hitchin correspondence**; Donaldson, Buchsdahl, Uhlenbeck-Yau, Li-Yau, Lübke-Teleman): Let B be a holomorphic vector bundle. **Then B admits a Yang-Mills metric if and only if B is polystable.**

COROLLARY: Any tensor product of polystable bundles is polystable.

REMARK: This result **was generalized to coherent sheaves** by Bando and Siu.

REMARK: **Stability is required** if you want to classify vector bundles or construct their moduli spaces.

Stable bundles on transversally Kähler manifolds

THEOREM: Let (M, ω) be a Vaisman manifold, $\dim M > 2$. Consider a vector bundle B with a Yang-Mills connection ∇, $\deg_\omega B = 0$. **Then the curvature Θ_B of ∇ satisfies $\Theta_B(x, \cdot) = 0$ for any $x \in \Sigma$.**

Proof: Yesterday's lecture.

COROLLARY: In these assumptions, let $X \in \Sigma$ be a holomorphic vector field tangent to the foliation Σ. **Then ∇_X takes holomorphic sections of B to holomorphic sections.** In particular, **it defines a holomorphic vector field on $\operatorname{Tot}(B)$ which projects to X.**

Proof: Yesterday's lecture.

YESTERDAY'S MAIN THEOREM: Let B be a stable coherent bundle on a quairegular Vaisman manifold, and let $\pi : M \longrightarrow X = M/\Sigma$ be the projection map. **Then $F = \pi^* B_0 \otimes L$, where B_0 is a coherent sheaf on M/Σ, and L a line bundle.**

Today's main question: What can be done when M is not quasiregular?

Equivariant structure on stable bundles

THEOREM 1: Let (M, Σ, ω_0) be a Vaisman manifold, $\dim M > 2$, B a stable line bundle of degree 0, and ∇ its Yang-Mills metric. Consider the holomorphic vector fields $\nabla_X \in T\operatorname{Tot}(B)$, $\nabla_{IX} \in T\operatorname{Tot}(B)$, denoted by $\tilde{X}$, $I(\tilde{X})$. **Then the corresponding diffeomorphism flow acts on $\operatorname{Tot}(B)$ by holomorphic isometries, and its closure, denoted by $G^\diamond$, is a compact Lie group.**

Proof: Since ∇_X commutes with $\overline{\partial}$, the corresponding vector field on the total space $\operatorname{Tot} B$ preserves holomorphic sections and acts on $\operatorname{Tot} B$ by holomorphic isometries. **Since this action preserves the length of a section, it acts on the corresponding sphere bundle,** and its closure is a compact Lie group.

■

Closure of the diffeomorphism flow generated by the Lee fields

THEOREM: Let M be a Vaisman manifold, and $\tilde{M} = C(X)$ its $\mathbb{Z}$-covering, identified with an algebraic cone $C(Z) = \mathrm{Tot}^\circ(L)$ over a Kähler orbifold Z. Denote by X its Lee field, and $G \subset \mathrm{Diff}(M)$ be the closure of the group of diffeomorphisms denerated by e^{tX} and $e^{tI(X)}$, $t \in \mathbb{R}$. **Then G is a compact torus**, acting on X by holomorphic Hamiltonian isometries.

Proof: G is compact because X and $I(X)$ are Killing, and acts by holomorphic automorphisms because X and $I(X)$ are holomorphic. It is Hamiltonian, because X and $I(X)$ are Hamiltonian vector fields. ■

THEOREM: In assumptions of this theorem, let $\tilde{G}$ be the lift of G to the cone $C(Z)$, and γ the generator of the $\mathbb{Z}$-action on $C(Z)$, such that $M := \frac{C(Z)}{\langle\gamma\rangle}$. **Then γ^n belongs to $\tilde{G}$ for some $n \in \mathbb{Z}^{>0}$.**

Proof: Principles of Locally Conformally Kähler Geometry, by Liviu Ornea, Misha Verbitsky.

The difference between G and $G^\diamond$

Let B be a stable bundle over a Vaisman manifold, $\deg B = 0$. Then B is $G^\diamond$-equivariant (Theorem 1), with $G^\diamond$ obtained by lifting the Lee vector field to $\operatorname{Tot}(B)$ and taking the closure.

CLAIM: The natural projection $u: G^\diamond \to G$ **is surjective, and its kernel is naturally embedded to $U(1)$.**

Proof: The groups G, $G^\diamond$ are compact, and obtained by taking the closure of the diffeomorphism flow of the Lee field; this implies the surjectivity. For any $g \in \ker u$, g acts by a Hermitian automorphism on B; since B is stable, $\operatorname{Aut}(B) = \mathbb{C}^*$, and its group of isometries is $U(1)$. ∎

COROLLARY: The group $G^\diamond$ **is isomorphic to a connected finite covering of G or to $G \times U(1)$.**

Proof: This group is compact, hence the kernel of the map $u: G^\diamond \to G$ is either $U(1)$ or a finite subgroup of $U(1)$. Also, $G^\diamond$ is connected, hence in the second case $G^\diamond$ is a finite connected covering. ∎

An idea: Passing to principal $PU(n)$-bundles, **we can forget about the difference between $G^\diamond$-equivariant structure and G-equivariant structure.**

Equivariance of $PU(n)$-bundles

Let M be a Vaisman manifold, and $\tilde{M} = C(X)$ its $\mathbb{Z}$-covering, with $M = \frac{C(Z)}{\langle\gamma\rangle}$. Assume that $\gamma \in \tilde{G}$. Clearly, vector bundles on M are the same as $\langle\gamma\rangle$-equivariant bundles on $\tilde{M}$. However, since $\langle\gamma\rangle \subset \tilde{G}$, **$\langle\gamma\rangle$-equivariancy is implied by $\tilde{G}$-equivariance.**

THEOREM: Every $PU(n)$-principal bundle equipped with a Yang-Mills metric on M **is obtained as a pullback of a $\tilde{G}$-equivariant Yang-Mills $PU(n)$-bundle on Z.** Conversely, **every $\tilde{G}$-equivariant Yang-Mills $PU(n)$-bundle on Z is pulled back to a $PU(n)$-principal Yang-Mills bundle on M.**

COROLLARY: Stable coherent sheaves on Vaisman manifolds **are filtrable,** that is, admit a filtration by coherent sheaves with subquotients of rank 1.