Ergodic complex structures

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**Ergodic complex structures**

**DEFINITION:** Let $M$ be a smooth manifold. **A complex structure** on $M$ is an endomorphism $I \in \text{End} \, TM$, $I^2 = -\text{Id}_{TM}$ such that the eigenspace bundles of $I$ are involutive, that is, satisfy $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$.

**REMARK:** Let Comp be the space of such tensors equipped with a topology of convergence of all derivatives. It is a Fréchet manifold.

**DEFINITION:** The diffeomorphism group Diff is a Fréchet Lie group acting on Comp in a natural way. A complex structure is called **ergodic** if its Diff-orbit is dense in Comp.

**REMARK:** The “moduli space” of complex structures (if it exists) is identified with Comp / Diff; existence of ergodic complex structures guarantees that the moduli space does not exist (all points are non-separable).

**THEOREM:** Let $M$ be a compact torus, $\dim_{\mathbb{C}} M \geq 2$, or a simple hyperkähler manifold. **A complex structure on $M$ is ergodic if and only if $\text{Pic}(M)$ is not of maximal rank.**
Period space as a Grassmannian of positive 2-planes

**PROPOSITION:** The period space

$$\text{Per} := \{ l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0 \}$$

is identified with $Gr_{+,-}(H^2(M, \mathbb{R})) = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$, which is a Grassmannian of positive oriented 2-planes in $H^2(M, \mathbb{R})$.

**Proof.** **Step 1:** Given $l \in \mathbb{P}H^2(M, \mathbb{C})$, the space generated by $\text{Im} \ l, \text{Re} \ l$ is 2-dimensional, because $q(l, l) = 0, q(l, \bar{l})$ implies that $l \cap H^2(M, \mathbb{R}) = 0$.

**Step 2:** This 2-dimensional plane is positive, because $q(\text{Re} \ l, \text{Re} \ l) = q(l + \bar{l}, l + \bar{l}) = 2q(l, \bar{l}) > 0$.

**Step 3:** Conversely, for any 2-dimensional positive plane $V \in H^2(M, \mathbb{R})$, the quadric $\{ l \in V \otimes_{\mathbb{R}} \mathbb{C} \mid q(l, l) = 0 \}$ consists of two lines; a choice of a line is determined by orientation. ■
Birational Teichmüller moduli space

**DEFINITION:** Let $M$ be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

**THEOREM:** (Huybrechts) Two points $I, I' \in \text{Teich}$ are non-separable if and only if there exists a bimeromorphism $(M, I) \to (M, I')$ which is non-singular in codimension 2.

**DEFINITION:** The space $\text{Teich}_b := \text{Teich} / \sim$ is called the **birational Teichmüller space** of $M$.

**THEOREM:** The period map $\text{Teich}_b \xrightarrow{\text{Per}} \text{Per}$ is an isomorphism, for each connected component of $\text{Teich}_b$.

**THEOREM:** Let $(M, I)$ be a hyperkähler manifold, and $W$ a connected component of its birational moduli space. Then the set of isomorphism classes of complex holomorphically symplectic structures is identified with $\text{Per}/\Gamma$, where $\text{Per} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ and $\Gamma$ is a finite index subgroup in $O(H^2(M, \mathbb{Z}), q)$, called the **monodromy group**.
**Ergodic complex structures**

**DEFINITION:** Let \((M, \mu)\) be a space with measure, and \(G\) a group acting on \(M\) preserving measure. This action is **ergodic** if all \(G\)-invariant measurable subsets \(M' \subset M\) satisfy \(\mu(M') = 0\) or \(\mu(M \setminus M') = 0\).

**CLAIM:** Let \(M\) be a manifold, \(\mu\) a Lebesgue measure, and \(G\) a group acting on \(M\) ergodically. **Then the set of non-dense orbits has measure 0.**

**Proof. Step 1:** Consider a non-empty open subset \(U \subset M\). Then \(\mu(U) > 0\), hence \(M' := G \cdot U\) satisfies \(\mu(M \setminus M') = 0\). For any orbit \(G \cdot x\) not intersecting \(U\), \(x \in M \setminus M'\). Therefore the set \(Z_U\) of such orbits has measure 0.

**Proof. Step 2:** Choose a countable base \(\{U_i\}\) of topology on \(M\). Then the set of points with dense orbits is \(M \setminus \bigcup_i Z_{U_i}\). \(\blacksquare\)

**DEFINITION:** Let \(M\) be a complex manifold, Teich its Teichmüller space, and \(\Gamma\) the mapping group acting on Teich **An ergodic complex structure** is a complex structure with dense \(\Gamma\)-orbit.
Ergodic complex structures

**Ergodicity of the monodromy group action**

**DEFINITION:** A lattice in a Lie group is a discrete subgroup $\Gamma \subset G$ such that $G/\Gamma$ has finite volume with respect to Haar measure.

**THEOREM:** (Calvin C. Moore, 1966) Let $\Gamma$ be a lattice in a non-compact simple Lie group $G$ with finite center, and $H \subset G$ a non-compact subgroup. Then the left action of $\Gamma$ on $G/H$ is ergodic.

**THEOREM:** Let $\text{Per}$ be a component of a birational Teichmüller space, and $\Gamma$ its monodromy group. Let $\text{Per}_e$ be a set of all points with dense orbits. Then $Z := \text{Per} \setminus \text{Per}_e$ has measure 0.

**Proof:** Let $G = SO(b_2 - 3, 3)$, $H = SO(2) \times SO(b_2 - 3, 1)$. Then $\Gamma_I$-action on $G/H = \text{Per}$ is ergodic, by Moore’s theorem. Therefore, all orbits outside of a measure 0 set are dense. ■

**REMARK:** Generic deformation of $M$ has no rational curves, and no non-trivial birational models. Therefore, outside of a measure zero subset, $\text{Teich} = \text{Teich}_b$. This implies that almost all complex structures on $M$ are ergodic.
Ratner's theorem

**DEFINITION:** Let $G$ be a connected Lie group equipped with a Haar measure. A lattice $\Gamma \subset G$ is a discrete subgroup of finite covolume (that is, $G/\Gamma$ has finite volume).

**REMARK:** Arithmetic lattices in simple Lie groups have finite covolume (Borel, Harish-Chandra).

**THEOREM:** Let $H \subset G$ be a Lie subgroup generated by unipotents, and $\Gamma \subset G$ an arithmetic lattice. Then a closure of any $H$-orbit in $G/\Gamma$ is an orbit of a closed, connected subgroup $S \subset G$, such that $S \cap \Gamma \subset S$ is a lattice.

**REMARK:** A closure of $H \cdot x$ in $G/\Gamma$ and a closure of $x \cdot \Gamma$ in $H \setminus G$ are related as follows. Let $\pi_1 : G \to H \setminus G$, $\pi_2 : G \to G/\Gamma$ be the projections. Then the connected component of $\pi_2^{-1}(H \cdot x_{G/\Gamma})$ is the connected component of $\pi_1^{-1}(x \cdot \Gamma_{H \setminus G})$.

**EXAMPLE:** Let $V$ be a real vector space with a non-degenerate bilinear symmetric form of signature $(3, k)$, $k > 0$, $G := SO^+(V)$ a connected component
of the isometry group, $H \subset G$ a subgroup fixing a given positive 2-dimensional plane, $H \cong SO^+(1, k) \times SO(2)$, and $\Gamma \subset G$ an arithmetic lattice. Consider the quotient $\mathbb{P}er := G/H$. Then a closure of $\Gamma \cdot J$ in $G/H$ is $\Gamma \cdot J \cdot S$, for some Lie subgroup $S \supset H$. Moreover, $S = H$ if and only if the orbit $\Gamma \cdot J$ is closed.
Characterization of ergodic complex structures

CLAIM: Let $G = SO^+(3,k)$, and $H \cong SO^+(1,k) \times SO(2) \subset G$. Then any closed connected Lie subgroup $S \subset G$ containing $H$ coincides with $G$ or with $H$.

COROLLARY: Let $J \in \text{Per} = G/H$. Then either $J$ is ergodic, or its $\Gamma$-orbit is closed in $\text{Per}$.

REMARK: By Ratner’s theorem, in the latter case the $H$-orbit of $J$ has finite volume in $G/\Gamma$. Therefore, its intersection with $\Gamma$ is a lattice in $H$. This brings

COROLLARY: Let $J \in \text{Per}$ be a point, such that its $\Gamma$-orbit is closed in $\text{Per}$. Consider its stabilizer $\text{St}(J) \cong H \subset G$. Then $\text{St}(J) \cap \Gamma$ is a lattice in $\text{St}(J)$.

COROLLARY: Let $J$ be a non-ergodic complex structure on a hyperkähler manifold, and $W \subset H^2(M,\mathbb{R})$ be a plane generated by $\text{Re} \Omega, \text{Im} \Omega$. Then $W$ is rational.

REMARK: This can be used to show that any hyperkähler manifold is Kobayashi non-hyperbolic.
**Kobayashi pseudometric**

**REMARK:** The results further on are from a joint paper arXiv:1308.5667 by Ljudmila Kamenova, Steven Lu, Misha Verbitsky.

**DEFINITION:** Pseudometric on $M$ is a function $d: M \times M \rightarrow R^{\geq 0}$ which is symmetric: $d(x,y) = d(y,x)$ and satisfies the triangle inequality $d(x,y) + d(y,z) \geq d(x,z)$.

**REMARK:** Let $\mathcal{D}$ be a set of pseudometrics. Then $d_{\text{max}}(x,y) := \sup_{d \in \mathcal{D}} d(x,y)$ is also a pseudometric.

**DEFINITION:** The **Kobayashi pseudometric** on a complex manifold $M$ is $d_{\text{max}}$ for the set $\mathcal{D}$ of all pseudometrics such that any holomorphic map from the Poincaré disk to $M$ is distance-non-increasing.

**THEOREM:** Let $\pi: M \rightarrow X$ be a smooth holomorphic family, which is trivialized as a smooth manifold: $M = M \times X$, and $d_x$ the Kobayashi metric on $\pi^{-1}(x)$. Then $d_x(m,m')$ is upper continuous on $x$.

**COROLLARY:** Denote the diameter of the Kobayashi pseudometric by $\text{diam}(d_x) := \sup_{m,m'} d_x(m,m')$. Then the Kobayashi diameter of a fiber of $\pi$ is an upper continuous function: $\text{diam} : X \rightarrow R^{\geq 0}$. 

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**Vanishing of Kobayashi pseudometric**

**THEOREM:** Let \((M, I)\) be a complex manifold with vanishing Kobayashi pseudometric. Then **the Kobayashi pseudometric vanishes for all ergodic complex structures in the same deformation class.**

**Proof:** Let \(\text{diam} : \text{Comp} \to \mathbb{R}^{\geq 0}\) map a complex structure \(J\) to the diameter of the Kobayashi pseudometric on \((M, J)\). Let \(J\) be an ergodic complex structure. The set of points \(J' = \nu(J) \in \text{Comp}\) such that \((M, J')\) is biholomorphic to \((M, J)\) is dense, because \(J\) is ergodic. By upper semi-continuity, \(0 = \text{diam}(I) \geq \inf_{J' = \nu(J)} \text{diam}(J)\). \(\blacksquare\)

**EXAMPLE:** Let \(M\) be a projective K3 surface. Then the Kobayashi metric on \(M\) vanishes. Since all non-projective K3 are ergodic, the Kobayashi metric vanishes on non-projective K3 surfaces as well.

**THEOREM:** Let \(M\) be a compact simple hyperkähler manifold. Assume that a deformation of \(M\) admits a holomorphic Lagrangian fibration and the Picard rank of \(M\) is not maximal. **Then the Kobayashi pseudometric on \(M\) vanishes.**

**THEOREM:** Let \(M\) be a Hilbert scheme of K3. **Then the Kobayashi pseudometric on \(M\) vanishes.**
Kobayashi hyperbolic manifolds

**DEFINITION:** An entire curve is a non-constant map $\mathbb{C} \to M$.

**DEFINITION:** A compact complex manifold $M$ is called **Kobayashi hyperbolic** if there exist no entire curves $\mathbb{C} \to M$.

**THEOREM:** (Brody, 1975)
Let $I_i$ be a sequence of complex structures on $M$ which are not hyperbolic, and $I$ its limit. Then $(M, I)$ is also not hyperbolic.

**THEOREM:** All hyperkähler manifolds are non-hyperbolic.

**REMARK:** This theorem would follow if we produce an ergodic complex structure which is non-hyperbolic. Indeed, a closure of its orbit is the whole Teich, and a limit of non-hyperbolic complex structures is non-hyperbolic.
**Twistor spaces and hyperkähler geometry**

**DEFINITION:** A hyperkähler structure on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**DEFINITION:** Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{ L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1 \}$

**DEFINITION:** A twistor space $\text{Tw}(M)$ of a hyperkähler manifold is a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{CP}^1$. More formally:

Let $\text{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_mM \to T_mM$ on $M$ induced by $J \in S^2 \subset \mathbb{H}$. Let $I_J$ denote the complex structure on $S^2 \cong \mathbb{CP}^1$.

The operator $I_{\text{Tw}} = I_m \oplus I_J : T_x \text{Tw}(M) \to T_x \text{Tw}(M)$ satisfies $I_{\text{Tw}}^2 = -\text{Id}$. **It defines an almost complex structure on $\text{Tw}(M)$**. This almost complex structure is known to be integrable (Obata).
Entire curves in twistor fibers

**THEOREM: (F. Campana, 1992)**
Let $M$ be a hyperkähler manifold, and $\text{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$ its twistor projection. **Then there exists an entire curve in some fiber of $\pi$.**

**CLAIM:** There exists a twistor family which has only ergodic fibers.

**Proof:** There are only countably many complex structures which are not ergodic. ■

**THEOREM:** All hyperkähler manifolds are non-hyperbolic.

**Proof:** Let $\text{Tw}(M) \rightarrow \mathbb{C}P^1$ be a twistor family with all fibers ergodic. **By Campana’s theorem, one of these fibers, denoted $(M, I)$, is non-hyperbolic.** Since any complex structure $I' \in \text{Teich}$ lies in the closure of $\text{Diff}(M) \cdot I$, all complex structures $I' \in \text{Teich}$ are non-hyperbolic. ■