Ergodic complex structures

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Ergodic complex structures

DEFINITION: Let M be a smooth manifold. **A complex structure** on M is an endomorphism $I \in \operatorname{End} TM$, $I^2 = -\operatorname{Id}_{TM}$ such that the eigenspace bundles of I are **involutive**, that is, satisfy $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$.

REMARK: Let Comp be the space of such tensors equipped with a topology of convergence of all derivatives. **It is a Fréchet manifold**.

DEFINITION: The diffeomorphism group Diff is a Fréchet Lie group acting on Comp in a natural way. A complex structure is called **ergodic** if its Diff-orbit is dense in Comp.

REMARK: The "moduli space" of complex structures (if it exists) is identified with Comp / Diff; **existence of ergodic complex structures guarantees that the moduli space does not exist** (all points are non-separable).

THEOREM: Let M be a compact torus, $\dim_{\mathbb{C}} M \geqslant 2$, or a simple hyperkähler manifold. **A complex structure on M is ergodic if and only if $\operatorname{Pic}(M)$ is not of maximal rank.**

Period space as a Grassmannian of positive 2-planes

PROPOSITION: The period space

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \ | \ \ q(l,l) = 0, q(l,\bar{l}) > 0.\}$$

is identified with $Gr_{+,+}(H^2(M,\mathbb{R})) = SO(b_2-3,3)/SO(2) \times SO(b_2-3,1)$**,** which is a Grassmannian of positive oriented 2-planes in $H^2(M,\mathbb{R})$.

Proof. Step 1: Given $l \in \mathbb{P}H^2(M,\mathbb{C})$, **the space generated by** $\operatorname{Im} l, \operatorname{Re} l$ **is 2-dimensional,** because $q(l,l) = 0, q(l,\bar{l})$ implies that $l \cap H^2(M,\mathbb{R}) = 0$.

Step 2: This 2-dimensional plane is positive, because $q(\operatorname{Re} l, \operatorname{Re} l) = q(l + \bar{l}, l + \bar{l}) = 2q(l,\bar{l}) > 0$.

Step 3: Conversely, for any 2-dimensional positive plane $V \in H^2(M,\mathbb{R})$, **the quadric** $\{l \in V \otimes_{\mathbb{R}} \mathbb{C} \ \ | \ \ q(l,l) = 0\}$ **consists of two lines;** a choice of a line is determined by orientation. ■

Birational Teichmüller moduli space

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (Huybrechts) Two points $I, I' \in$ Teich **are non-separable if and only if there exists a bimeromorphism $(M, I) \longrightarrow (M, I')$ which is non-singular in codimension 2.**

DEFINITION: The space $\text{Teich}_b := \text{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\text{Teich}_b \xrightarrow{\text{Per}} \mathbb{P}\text{er}$ is an isomorphism, for each connected component of Teich_b.

THEOREM: Let (M, I) be a hyperkähler manifold, and W a connected component of its birational moduli space. **Then the set of isomorphism classes of complex holomorphically symplectic structures is identified with $\mathbb{P}\text{er}/\Gamma$, where $\mathbb{P}\text{er} = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$ and Γ is a finite index subgroup in $O(H^2(M,\mathbb{Z}), q)$**, called **the monodromy group**.

Ergodic complex structures

DEFINITION: Let (M, μ) be a space with measure, and G a group acting on M preserving measure. This action is **ergodic** if all G-invariant measurable subsets $M' \subset M$ satisfy $\mu(M') = 0$ or $\mu(M \backslash M') = 0$.

CLAIM: Let M be a manifold, μ a Lebesgue measure, and G a group acting on M ergodically. **Then the set of non-dense orbits has measure 0.**

Proof. Step 1: Consider a non-empty open subset $U \subset M$. Then $\mu(U) > 0$, hence $M' := G \cdot U$ satisfies $\mu(M \backslash M') = 0$. For any orbit $G \cdot x$ not intersecting U, $x \in M \backslash M'$. Therefore the set Z_U of such orbits has measure 0.

Proof. Step 2: Choose a countable base $\{U_i\}$ of topology on M. Then the set of points with dense orbits is $M \backslash \bigcup_i Z_{U_i}$. ∎

DEFINITION: Let M be a complex manifold, Teich its Techmüller space, and Γ the mapping group acting on Teich **An ergodic complex structure** is a complex structure with dense Γ-orbit.

Ergodicity of the monodromy group action

DEFINITION: A lattice in a Lie group is a discrete subgroup $\Gamma \subset G$ such that G/Γ has finite volume with respect to Haar measure.

THEOREM: (Calvin C. Moore, 1966) Let Γ be a lattice in a non-compact simple Lie group G with finite center, and $H \subset G$ a non-compact subgroup. **Then the left action of Γ on G/H is ergodic.**

THEOREM: Let $\mathbb{P}\mathrm{er}$ be a component of a birational Teichmüller space, and Γ its monodromy group. Let $\mathbb{P}\mathrm{er}_e$ be a set of all points with dense orbits. **Then $Z := \mathbb{P}\mathrm{er} \backslash \mathbb{P}\mathrm{er}_e$ has measure 0.**

Proof: Let $G = SO(b_2 - 3, 3)$, $H = SO(2) \times SO(b_2 - 3, 1)$. **Then Γ_I-action on $G/H = \mathbb{P}\mathrm{er}$ is ergodic,** by Moore's theorem. Therefore, all orbits outside of a measure 0 set are dense. ■

REMARK: Generic deformation of M has no rational curves, and no non-trivial birational models. Therefore, **outside of a measure zero subset,** $\mathrm{Teich} = \mathrm{Teich}_b$. This implies that **almost all complex structures on M are ergodic.**

Ratner's theorem

DEFINITION: Let G be a connected Lie group equipped with a Haar measure. **A lattice** $\Gamma \subset G$ is a discrete subgroup of finite covolume (that is, G/Γ has finite volume).

REMARK: Arithmetic lattices in simple Lie groups have finite covolume (Borel, Harish-Chandra).

THEOREM: Let $H \subset G$ be a Lie subroup generated by unipotents, and $\Gamma \subset G$ an arithmetic lattice. Then **a closure of any H-orbit in G/Γ is an orbit of a closed, connected subgroup $S \subset G$, such that $S \cap \Gamma \subset S$ is a lattice.**

REMARK: A closure of $H \cdot x$ in G/Γ and a closure of $x \cdot \Gamma$ in $H\backslash G$ are related as follows. Let $\pi_1 : G \longrightarrow H\backslash G$, $\pi_2 : G \longrightarrow G/\Gamma$ be the projections. Then the connected component of $\pi_2^{-1}(\overline{H \cdot x}_{G/\Gamma})$ is the connected component of $\pi_1^{-1}(\overline{x \cdot \Gamma}_{H\backslash G})$.

EXAMPLE: Let V be a real vector space with a non-degenerate bilinear symmetric form of signature $(3, k)$, $k > 0$, $G := SO^+(V)$ a connected component of the isometry group, $H \subset G$ a subgroup fixing a given positive 2-dimensional plane, $H \cong SO^+(1,k) \times SO(2)$, and $\Gamma \subset G$ an arithmetic lattice. Consider the quotient $\mathbb{P}\mathrm{er} := G/H$. **Then a closure of $\Gamma \cdot J$ in G/H is $\Gamma \cdot J \cdot S$, for some Lie subgroup $S \supset H$.** Moreover, $S = H$ if and only if the orbit $\Gamma \cdot J$ is closed.

Characterization of ergodic complex structures

CLAIM: Let $G = SO^+(3,k)$, and $H \cong SO^+(1,k) \times SO(2) \subset G$. Then **any closed connected Lie subgroup $S \subset G$ containing H coincides with G or with H.**

COROLLARY: Let $J \in \mathbb{P}\mathrm{er} = G/H$. Then **either J is ergodic, or its Γ-orbit is closed in $\mathbb{P}\mathrm{er}$.**

REMARK: By Ratner's theorem, in the latter case the H-orbit of J has finite volume in G/Γ. Therefore, **its intersection with Γ is a lattice in H.** This brings

COROLLARY: Let $J \in \mathbb{P}\mathrm{er}$ be a point, such that its Γ-orbit is closed in $\mathbb{P}\mathrm{er}$. Consider its stabilizer $\mathrm{St}(J) \cong H \subset G$. **Then $\mathrm{St}(J) \cap \Gamma$ is a lattice in $\mathrm{St}(J)$.**

COROLLARY: Let J be a non-ergodic complex structure on a hyperkähler manifold, and $W \subset H^2(M,\mathbb{R})$ be a plane generated by $\mathrm{Re}\,\Omega, \mathrm{Im}\,\Omega$. **Then W is rational.**

REMARK: This can be used to show that **any hyperkähler manifold is Kobayashi non-hyperbolic.**

Kobayashi pseudometric

REMARK: The results further on are from a joint paper arXiv:1308.5667 by Ljudmila Kamenova, Steven Lu, Misha Verbitsky.

DEFINITION: **Pseudometric** on M is a function $d: M \times M \longrightarrow \mathbb{R}^{\geqslant 0}$ which is symmetric: $d(x,y) = d(y,x)$ and satisfies the triangle inequality $d(x,y) + d(y,z) \geqslant d(x,z)$.

REMARK: Let $\mathfrak{D}$ be a set of pseudometrics. **Then** $d_{\max}(x,y) := \sup_{d\in\mathfrak{D}} d(x,y)$ **is also a pseudometric.**

DEFINITION: The **Kobayashi pseudometric** on a complex manifold M is $d_{\max}$ for the set $\mathfrak{D}$ of all pseudometrics such that any holomorphic map from the Poincaré disk to M is distance-non-increasing.

THEOREM: Let $\pi: \mathcal{M} \longrightarrow X$ be a smooth holomorphic family, which is trivialized as a smooth manifold: $\mathcal{M} = M \times X$, and d_x the Kobayashi metric on $\pi^{-1}(x)$. **Then** $d_x(m,m')$ **is upper continuous on** x**.** ■

COROLLARY: Denote the diameter of the Kobayashi pseudometric by $\operatorname{diam}(d_x) := \sup_{m,m'} d_x(m,m')$. **Then the Kobayashi diameter of a fiber of** π **is an upper continuous function:** $\operatorname{diam}: X \longrightarrow \mathbb{R}^{\geqslant 0}$**.**

Vanishing of Kobayashi pseudometric

THEOREM: Let (M, I) be a complex manifold with vanishing Kobayashi pseudometric. Then **the Kobayashi pseudometric vanishes for all ergodic complex structures in the same deformation class.**

Proof: Let diam : Comp $\longrightarrow \mathbb{R}^{\geqslant 0}$ map a complex structure J to the diameter of the Kobayashi pseudometric on (M, J). Let J be an ergodic complex structure. The set of points $J' = \nu(J) \in$ Comp such that (M, J') is biholomorphic to (M, J) is dense, because J is ergodic. By upper semi-continuity, $0 = \operatorname{diam}(I) \geqslant \inf_{J'=\nu(J)} \operatorname{diam}(J)$. ∎

EXAMPLE: Let M be a projective K3 surface. Then the Kobayashi metric on M vanishes. **Since all non-projective K3 are ergodic,** the Kobayashi metric vanishes on non-projective K3 surfaces as well.

THEOREM: Let M be a compact simple hyperkähler manifold. Assume that a deformation of M admits a holomorphic Lagrangian fibration and the Picard rank of M is not maximal. **Then the Kobayashi pseudometric on M vanishes.**

THEOREM: Let M be a Hilbert scheme of K3. **Then the Kobayashi pseudometric on M vanishes.**

Kobayashi hyperbolic manifolds

DEFINITION: An entire curve is a non-constant map $\mathbb{C} \longrightarrow M$.

DEFINITION: A compact complex manifold M is called **Kobayashi hyperbolic** if there exist no entire curves $\mathbb{C} \longrightarrow M$.

THEOREM: (Brody, 1975)
Let I_i be a sequence of complex structures on M which are not hyperbolic, and I its limit. Then (M, I) is also not hyperbolic.

THEOREM: All hyperkähler manifolds are non-hyperbolic.

REMARK: This theorem would follow if we produce an ergodic complex structure which is non-hyperbolic. Indeed, a closure of its orbit is the whole Teich, and a limit of non-hyperbolic complex structures is non-hyperbolic.

Twistor spaces and hyperkähler geometry

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

DEFINITION: **Induced complex structures** on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$

DEFINITION: A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^2 = -\operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata).

Entire curves in twistor fibers

THEOREM: (F. Campana, 1992)
Let M be a hyperkähler manifold, and $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$ its twistor projection. **Then there exists an entire curve in some fiber of π.**

CLAIM: There exists a twistor family which has only ergodic fibers.

Proof: There are only countably many complex structures which are not ergodic. ■

THEOREM: All hyperkähler manifolds are non-hyperbolic.

Proof: Let $\operatorname{Tw}(M) \longrightarrow \mathbb{C}P^1$ be a twistor family with all fibers ergodic. **By Campana's theorem, one of these fibers, denoted (M, I), is non-hyperbolic.** Since any complex structure $I' \in$ Teich lies in the closure of $\operatorname{Diff}(M) \cdot I$, all complex structures $I' \in$ Teich are non-hyperbolic. ■