Global Torelli theorem for hyperkähler manifolds

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Hyperkähler manifolds

**DEFINITION:** A hyperkähler structure on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**REMARK:** A hyperkähler manifold has three symplectic forms $\omega_I := g(I\cdot, \cdot), \omega_J := g(J\cdot, \cdot), \omega_K := g(K\cdot, \cdot)$.

**REMARK:** This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves $I, J, K$.

**DEFINITION:** Let $M$ be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called the holonomy group of $M$.

**REMARK:** A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving $I, J, K$).
**Holomorphically symplectic manifolds**

**DEFINITION:** A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

**REMARK:** Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1} \omega_K$ is a holomorphic symplectic form on $(M, I)$.

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

**DEFINITION:** For the rest of this talk, a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.

**DEFINITION:** A hyperkähler manifold $M$ is called simple if $H^1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

**Bogomolov’s decomposition:** Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

**Remark:** A simple hyperkähler manifold is always simply connected (Cheeger-Gromoll theorem).

**Further on, all hyperkähler manifolds are assumed to be simple.**
The Teichmüller space and the mapping class group

**Definition:** Let $M$ be a compact complex manifold, and $\text{Diff}_0 \subset \text{Diff}$ a connected component of its diffeomorphism group (the group of isotopies). Denote by $\text{Comp}$ the space of complex structures on $M$, and let $\text{Teich} := \text{Comp} / \text{Diff}_0$. We call it the **Teichmüller space**.

**Remark:** $\text{Teich}$ is a finite-dimensional complex space (Kodaira), but often non-Hausdorff.

**Definition:** We call $\Gamma := \text{Diff} / \text{Diff}_0$ the mapping class group. The moduli space of complex structures on $M$ is a connected component of $\text{Teich} / \Gamma$, but this quotient is not always well-defined.

**Remark:** This terminology is standard for curves.

**REMARK:** For hyperkähler manifolds, it is convenient to take for $\text{Teich}$ the space of all complex structures of hyperkähler type, that is, holomorphically symplectic and Kähler. It is open in the usual Teichmüller space.

**REMARK:** Two ingredients of global Torelli theorem:
(a) determine the mapping class group
(b) determine the Teichmüller space.
The Bogomolov-Beauville-Fujiki form

**Theorem:** (Fujiki). Let \( \eta \in H^2(M) \), and \( \dim M = 2n \), where \( M \) is hyperkähler. Then \( \int_M \eta^{2n} = cq(\eta, \eta)^n \), for some primitive integer quadratic form \( q \) on \( H^2(M, \mathbb{Z}) \), and \( c > 0 \) an integer number.

**Definition:** This form is called **Bogomolov-Beauville-Fujiki form**. It is defined by the Fujiki’s relation uniquely, up to a sign. The sign is determined from the following formula (Bogomolov, Beauville)

\[
\lambda q(\eta, \eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \frac{n-1}{n} \left( \int_X \eta \wedge \Omega^{n-1} \wedge \Omega \right) \left( \int_X \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1} \right)
\]

where \( \Omega \) is the holomorphic symplectic form, and \( \lambda > 0 \).

**Remark:** \( q \) has signature \((b_2 - 3, 3)\). It is negative definite on primitive forms, and positive definite on \( \langle \Omega, \overline{\Omega}, \omega \rangle \), where \( \omega \) is a Kähler form.
**Global Torelli Theorem**

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**Lefschetz \( \mathfrak{sl}(2) \)-action**

Let \((M, I, g)\) be a Kaehler manifold, \(\omega\) its Kaehler form. Consider the following operators.

1. \(L(\alpha) := \omega \wedge \alpha\).
2. \(\Lambda := L^*\) (Hermitian dual of \(L\)).
3. The Weil operator \(W_I|_{\Lambda^{p,q}(M)} = \sqrt{-1} (p - q)\)

**CLAIM:** The triple \(L, \Lambda, H\) satisfies the relations for the \(\mathfrak{sl}(2)\) Lie algebra:
\[
[L, \Lambda] = H, \quad [H, L] = 2L, \quad [H, \Lambda] = 2\Lambda.
\]

**DEFINITION:** \(L, \Lambda, H\) is called the Lefschetz \(\mathfrak{sl}(2)\)-triple.

**THEOREM:** The \(\mathfrak{sl}(2)\)-action \(\langle L, \Lambda, H \rangle\) and the action of Weil operator commute with Laplacian, hence preserve the harmonic forms on a Kähler manifold.

**COROLLARY:** Any cohomology class can be represented as a sum of closed \((p, q)\)-forms, **giving a decomposition** \(H^i(M) = \bigoplus_{p+q=i} H^{p,q}(M)\), with \(H^{p,q}(M) = H^{q,p}(M)\).
Riemann-Hodge pairing

THEOREM: Let \((M, \omega)\) be a Kähler \(n\)-manifold. Consider the following pseudo-Hermitian form on \(H^d(M)\):

\[
B(\alpha, \beta) := \int_M \alpha \wedge \overline{\beta} \wedge \omega^{n-d}
\]

Then \(B\) is sign-definite on a space \(H_{k}^{p,q}(M)\) of \((p, q)\)-forms which have weight \(k\) with respect to Lefschetz \(SL(2)\)-action.

COROLLARY: Let \(G\) be a group of automorphisms of the algebra \(H^*(M, \mathbb{R})\) preserving the \((p, q)\)-decomposition and fixing a Kähler class \(\omega\). Then \(G\) is compact.

Proof: Since \(G\) fixes \(\omega\), \(G\) commutes with the Lefschetz \(SL(2)\)-action, hence it fixes a sign-definite form on each space \(H_{k}^{p,q}(M)\). ■
**Automorphisms of cohomology.**

**THEOREM:** Let $M$ be a simple hyperkähler manifold, and $G \subset GL(H^*(M))$ a group of automorphisms of its cohomology algebra preserving the Pontryagin classes. Then $G$ acts on $H^2(M)$ **preserving the BBF form.** Moreover, the map $G \rightarrow O(H^2(M, \mathbb{R}), q)$ is surjective on a connected component, and has compact kernel.

**Proof.** **Step 1:** Fujiki formula $v^{2n} = q(v, v)^n$ implies that $\Gamma_0$ preserves the Bogomolov-Beauville-Fujiki up to a sign. The sign is fixed, if $n$ is odd.

**Step 2:** For even $n$, the sign is also fixed. Indeed, $G$ preserves $p_1(M)$, and (as Fujiki has shown) $v^{2n-2} \wedge p_1(M) = q(v, v)^{n-1}c$, for some $c \in \mathbb{R}$. The constant $c$ is positive, because the degree of $c_2(B)$ is positive for any non-trivial stable bundle with $c_1(B) = 0$.

**Step 3:** $O(H^2(M, \mathbb{R}), q)$ acts on $H^*(M, \mathbb{R})$ by derivations preserving Pontryagin classes (V., 1995). Therefore $\text{Lie}(G)$ surjects to $O(H^2(M, \mathbb{R}), q)$.

**Step 4:** The kernel $K$ of the map $G \rightarrow G|_{H^2(M, \mathbb{R})}$ is compact, because it commutes with the Hodge decomposition and Lefschetz $sl(2)$-action, hence preserves the Riemann-Hodge form.
Sullivan’s theorem

**Theorem:** (Sullivan) Let $M$ be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geq 3$. Denote by $\Gamma_0$ the group of automorphisms of an algebra $H^*(M, \mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then the natural map
\[ \text{Diff}(M)/\text{Diff}_0 \longrightarrow \Gamma_0 \] has finite kernel, and its image has finite index in $\Gamma_0$.

**Theorem:** Let $M$ be a simple hyperkähler manifold, and $\Gamma_0$ as above. Then
(i) $\Gamma_0 \big|_{H^2(M, \mathbb{Z})}$ is a finite index subgroup of $O(H^2(M, \mathbb{Z}), q)$.
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M, \mathbb{Z}), q)$ has finite kernel.

**Proof:** Follows from the computation of $G = \text{Aut}(H^*(M, \mathbb{R}), p_1, ..., p_n)$ done earlier. Indeed, the kernel of $\Gamma_0 \big|_{H^2(M, \mathbb{Z})}$ is a set of integer points of a compact Lie group, hence finite. The image of $\Gamma_0 = G_{\mathbb{Z}}$ has finite index in $O(H^2(M, \mathbb{Z}), q)$, because the corresponding map of Lie groups is surjective. ■
Computation of the mapping class group

**COROLLARY:** The mapping class group $\Gamma$ is mapped to $O(H^2(M,\mathbb{Z}),q)$ with finite kernel and finite index.

**Proof:** By Sullivan, $\Gamma$ is mapped to $\Gamma_0$ with finite kernel and finite index, and $\Gamma_0 \rightarrow O(H^2(M,\mathbb{Z}),q)$ has finite kernel and finite index, as shown above. ■

**THEOREM:** (Kollar-Matsusaka, Huybrechts) There are only finitely many connected components of Teich.

**COROLLARY:** Let $\Gamma_I$ be the group of elements of mapping class group preserving a connected component of Teichmüller space containing $I \in \text{Teich}$. Then $\Gamma_I$ has finite index in $\Gamma$.

**REMARK:** $\Gamma_I$ is a group generated by monodromy of all Gauss-Manin local systems for all deformations of $(M,I)$. It is known as the monodromy group of $(M,I)$. 
The period map

Remark: For any $J \in \text{Teich}$, $(M, J)$ is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let $P : \text{Teich} \rightarrow \mathbb{P}H^{2}(M, \mathbb{C})$ map $J$ to a line $H^{2,0}(M, J) \in \mathbb{P}H^{2}(M, \mathbb{C})$. The map $P : \text{Teich} \rightarrow \mathbb{P}H^{2}(M, \mathbb{C})$ is called the period map.

Remark: $P$ maps Teich into an open subset of a quadric, defined by

$$\text{Per} := \{ l \in \mathbb{P}H^{2}(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0 \}.$$ 

It is called the period space of $M$.

Remark: $\text{Per} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$

Theorem: Let $M$ be a simple hyperkähler manifold, and Teich its Teichmüller space. Then

(i) (Bogomolov) The period map $P : \text{Teich} \rightarrow \text{Per}$ is etale.
(ii) (Huybrechts) It is surjective.

Remark: Bogomolov’s theorem implies that Teich is smooth. It is usually non-Hausdorff.
Birational equivalence and non-separable points

**DEFINITION:** Let $M$ be a topological space. We say that $x, y \in M$ are non-separable (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

**THEOREM:** (D. Huybrechts) If $I_1, I_2 \in \text{Teich}$ are non-separable points, then $P(I_1) = P(I_2)$, and $(M, I_1)$ is birationally equivalent to $(M, I_2)$.

**COROLLARY:** The set of all non-separable points in $\text{Teich}$ belongs to a union of countably many divisors.

**Proof. Step 1:** Let $z \in H^2(M, \mathbb{Z})$ be a cohomology class, and $\text{Teich}_z$ the set of all $I \in \text{Teich}$ such that $z \in H^{1,1}(M, I)$. The relation $z \in H^{1,1}(M, I)$ is equivalent to $q(z, \Omega) = 0$, hence

$$\text{Per}(\text{Teich}_z) \subset \{l \in \mathbb{P}(z^\perp) \mid q(l, l) = 0, q(l, I) > 0\}.$$  

This shows that $\text{Per}(\text{Teich}_z)$ is a divisor.

**Step 2:** If $(M, I)$ is a holomorphic symplectic manifold with non-trivial birational equivalence to $M'$, it contains a rational curve $C$. Therefore, $I \in \text{Teich}_z$, where $z = [C]$ is its homology class. ■
Hausdorff reduction

**REMARK:** A non-Hausdorff manifold is a topological space locally diffeomorphic to $\mathbb{R}^n$.

**DEFINITION:** Let $M$ be a topological space for which $M/\sim$ is Hausdorff. Then $M/\sim$ is called a Hausdorff reduction of $M$.

**Problems:**
1. $\sim$ is not always an equivalence relation.
2. Even if $\sim$ is equivalence, the $M/\sim$ is not always Hausdorff.

**REMARK:** A quotient $M/\sim$ is Hausdorff, if $M \to M/\sim$ is open, and the graph $\Gamma\sim \in M \times M$ is closed.
Weakly Hausdorff manifolds

**DEFINITION:** A point \( x \in X \) is called **Hausdorff** if \( x \not\sim y \) for any \( y \neq x \).

**DEFINITION:** Let \( M \) be an \( n \)-dimensional real analytic manifold, not necessarily Hausdorff. Suppose that the set \( Z \subseteq M \) of non-Hausdorff points is contained in a countable union of real analytic subvarieties of codim \( \geq 2 \). Suppose, moreover, that

(S) For every \( x \in M \), there is a closed neighbourhood \( B \subseteq M \) of \( x \) and a continuous surjective map \( \Psi : B \rightarrow \mathbb{R}^n \) to a closed ball in \( \mathbb{R}^n \), **inducing a homeomorphism** on an open neighbourhood of \( x \).

Then \( M \) is called a **weakly Hausdorff manifold**.

**REMARK:** The period map satisfies (S). Also, the non-Hausdorff points of Teich are contained in a countable union of divisors.

**THEOREM:** A weakly Hausdorff manifold \( X \) admits a Hausdorff reduction. In other words, the quotient \( X/\sim \) is a Hausdorff. Moreover, \( X \rightarrow X/\sim \) is locally a homeomorphism.

This theorem is proven using 1920-ies style point-set topology.
Birational Teichmüller moduli space

**DEFINITION:** The space \( \text{Teich}_b := \text{Teich} / \sim \) is called the birational Teichmüller space of \( M \).

**THEOREM:** The period map \( \text{Teich}_b \xrightarrow{\text{Per}} \text{Per} \) is an isomorphism, for each connected component of \( \text{Teich}_b \).

The proof is based on two results.

**PROPOSITION:** (The Covering Criterion) Let \( X \xrightarrow{\varphi} Y \) be an etale map of smooth manifolds. Suppose that each \( y \in Y \) has a neighbourhood \( B \ni y \) diffeomorphic to a closed ball, such that for each connected component \( B' \subset \varphi^{-1}(B) \), \( B' \) projects to \( B \) surjectively. **Then \( \varphi \) is a covering.**

**PROPOSITION:** The period map satisfies the conditions of the Covering Criterion.
Global Torelli theorem

**DEFINITION:** Let $M$ be a hyperkaehler manifold, $\text{Teich}_b$ its birational Teichmüller space, and $\Gamma$ the mapping class group. The quotient $\text{Teich}_b/\Gamma$ is called the **birational moduli space** of $M$.

**REMARK:** The birational moduli space is obtained from the usual moduli space by gluing some (but not all) non-separable points. **It is still non-Hausdorff.**

**THEOREM:** Let $(M, I)$ be a hyperkähler manifold, and $W$ a connected component of its birational moduli space. Then $W$ is isomorphic to $\text{Per}/\Gamma_I$, where $\text{Per} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ and $\Gamma_I$ is an arithmetic subgroup in $O(H^2(M, \mathbb{R}), q)$.

**A CAUTION:** Usually “the global Torelli theorem” is understood as a theorem about Hodge structures. For K3 surfaces, the Hodge structure on $H^2(M, \mathbb{Z})$ determines the complex structure. For $\dim_{\mathbb{C}} M > 2$, it is false.
The marked moduli space

**DEFINITION:** Let Γ be the mapping class group, and \( K \subset \Gamma \) the kernel of the natural map \( \Gamma \rightarrow GL(H^2(M,\mathbb{Z})) \). **It is finite,** as we have shown. The quotient \( \text{Teich}/K \) is called the marked moduli space.

**THEOREM:** The natural map \( \text{Teich} \rightarrow \text{Teich}/K \) is a homeomorphism on each connected component.

**Proof. Step 1:**
Let \( I \in \text{Teich} \) be a fixed point of a subgroup \( K_I \subset K \). By Bogomolov’s theorem, \( T_I \text{Teich} \) is naturally identified with \( H^{1,1}_I(M) \). Since the action of \( K \) on \( H^2(M) \) is trivial, any \( \alpha \in K_I \) acts trivially on \( T_I \text{Teich} \). Therefore, \( K_I \) acts as identity on a connected component of \( \text{Teich} \) containing \( I \).

**Step 2:** From Step 1, obtain that the quotient map \( \text{Teich} \xrightarrow{\Psi} \text{Teich}/K \) is a finite covering, hence it induces a finite covering of the corresponding Hausdorff reductions. However, \( \Psi \) induces an isomorphism on each connected component of \( \text{Teich}_b \), because each component of \( \text{Teich}_b \) is isomorphic to \( \mathbb{P}_{\text{er}} \). ■
The Hodge-theoretic Torelli theorem

**REMARK:** The group $O(p, q)$ $(p, q > 0)$ has 4 connected components, corresponding to the orientations of positive $p$-dimensional and negative $q$-dimensional planes.

**DEFINITION:** Let $M$ be a hyperkaehler manifold. One says that the Hodge-theoretic Torelli theorem holds for $M$ if

$$\text{Teich}/\Gamma_I \longrightarrow \mathbb{P}er/O^+(H^2(M, \mathbb{Z}), q),$$

where $O^+(H^2(M, \mathbb{Z}), q)$ is a subgroup of $O(H^2(M, \mathbb{Z}), q)$ preserving orientation on positive 3-planes. Equivalently, it is true if $M$ is uniquely determined by its Hodge structure.

**REMARK:** The Hodge-theoretic Torelli theorem is true for K3 surfaces. It is false for all other known examples of hyperkaehler manifolds.

**Problems:**
1. The moduli space $\text{Teich}/\Gamma$ is not Hausdorff (Debarre, 1984). Indeed, bimeromorphically equivalent hyperkähler manifolds have isomorphic Hodge structures.
2. The covering $\text{Teich}_b/\Gamma_I \longrightarrow \mathbb{P}er/O^+(H^2(M, \mathbb{Z}), q)$ is non-trivial, because the map $\Gamma_I \longrightarrow O^+(H^2(M, \mathbb{Z}), q)$ is not surjective (Namikawa, 2002).
The birational Hodge-theoretic Torelli theorem

**DEFINITION:** The birational Hodge-theoretic Torelli theorem is true for $M$ if $\Gamma_I$ (the stabilizer of a Torelli component in the mapping class group) is isomorphic to $O^+(H^2(M, \mathbb{Z}), q)$.

**REMARK:** If a birational Hodge-theoretic Torelli theorem holds for $M$, then any deformation of $M$ is up to a bimeromorphic equivalence determined by the Hodge structure on $H^2(M)$.

**THEOREM:** (Markman) The for $M = K3^{[n]}$, the group $\Gamma_I$ is a subgroup of $O^+(H^2(M, \mathbb{Z}), q)$ generated by oriented reflections.

**THEOREM:** Let $M = K3^{[n+1]}$ with $n$ a prime power. Then the (usual) global Torelli theorem holds birationally: two deformations of a Hilbert scheme with isomorphic Hodge structures are bimeromorphic. For other $n$, it is false (Markman).