Global Torelli theorem for hyperkähler manifolds

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Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $H^1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Remark: A simple hyperkähler manifold is always simply connected (Cheeger-Gromoll theorem).

Further on, all hyperkähler manifolds are assumed to be simple.

The Teichmüller space and the mapping class group

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0 \subset \operatorname{Diff}$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Comp the space of complex structures on M, and let $\operatorname{Teich} := \operatorname{Comp}/\operatorname{Diff}_0$. We call it **the Teichmüller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira), but often **non-Hausdorff**.

Definition: We call $\Gamma := \operatorname{Diff}/\operatorname{Diff}_0$ **the mapping class group**. The **moduli space of complex structures on M** is a connected component of $\operatorname{Teich}/\Gamma$, but this quotient is not always well-defined.

Remark: This terminology is **standard for curves.**

REMARK: For hyperkähler manifolds, it is convenient to take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler**. It is open in the usual Teichmüller space.

REMARK: Two ingredients of global Torelli theorem:
(a) determine the mapping class group
(b) determine the Teichmüller space.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(b_2 - 3, 3)$.** It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

Lefschetz $\mathfrak{sl}(2)$-action

Let (M, I, g) be a Kaehler manifold, ω its Kaehler form. Consider the following operators.

1. $L(\alpha) := \omega \wedge \alpha$.
2. $\Lambda := L^*$ (Hermitian dual of L).
3. The Weil operator $W_I\Big|_{\Lambda^{p,q}(M)} = \sqrt{-1}\,(p-q)$

CLAIM: **The triple L, Λ, H satisfies the relations for the $\mathfrak{sl}(2)$ Lie algebra:** $[L, \Lambda] = H$, $[H, L] = 2L$, $[H, \Lambda] = 2\Lambda$.

DEFINITION: L, Λ, H is called **the Lefschetz $\mathfrak{sl}(2)$-triple**.

THEOREM: The $\mathfrak{s}l(2)$-action $\langle L, \Lambda, H\rangle$ and the action of Weil operator commute with Laplacian, hence **preserve the harmonic forms on a Kähler manifold**.

COROLLARY: Any cohomology class can be represented as a sum of closed (p,q)-forms, **giving a decomposition $H^i(M) = \bigoplus_{p+q=i} H^{p,q}(M)$, with $\overline{H^{p,q}(M)} = H^{q,p}(M)$.**

Riemann-Hodge pairing

THEOREM: Let (M, ω) be a Kähler n-manifold. Consider the following pseudo-Hermitian form on $H^d(M)$:

$$B(\alpha, \beta) := \int_M \alpha \wedge \overline{\beta} \wedge \omega^{n-d}$$

Then B is sign-definite on a space $H_k^{p,q}(M)$ of (p,q)-forms which have weight k with respect to Lefschetz $SL(2)$-action.

COROLLARY: Let G be a group of automorphisms of the algebra $H^*(M, \mathbb{R})$ preserving the (p,q)-decomposition and fixing a Kähler class ω. **Then G is compact.**

Proof: Since G fixes ω, G commutes with the Lefschetz $SL(2)$-action, hence it fixes a sign-definite form on each space $H_k^{p,q}(M)$. ■

Automorphisms of cohomology.

THEOREM: Let M be a simple hyperkähler manifold, and $G \subset GL(H^*(M))$ a group of automorphisms of its cohomology algebra preserving the Pontryagin classes. Then G acts on $H^2(M)$ **preserving the BBF form.** Moreover, the map $G \longrightarrow O(H^2(M,\mathbb{R}),q)$ **is surjective on a connected component, and has compact kernel.**

Proof. Step 1: Fujiki formula $v^{2n} = q(v,v)^n$ implies that Γ_0 **preserves the Bogomolov-Beauville-Fujiki up to a sign.** The sign is fixed, if n is odd.

Step 2: For even n, the sign is also fixed. Indeed, G preserves $p_1(M)$, and (as Fujiki has shown) $v^{2n-2} \wedge p_1(M) = q(v,v)^{n-1}c$, for some $c \in \mathbb{R}$. The constant c is positive, **because the degree of $c_2(B)$ is positive** for any non-trivial stable bundle with $c_1(B) = 0$.

Step 3: $\mathfrak{o}(H^2(M,\mathbb{R}),q)$ acts on $H^*(M,\mathbb{R})$ by derivations preserving Pontryagin classes (V., 1995). Therefore Lie(G) surjects to $\mathfrak{o}(H^2(M,\mathbb{R}),q)$.

Step 4: The kernel K of the map $G \longrightarrow G\big|_{H^2(M,\mathbb{R})}$ is compact, because it commutes with the Hodge decomposition and Lefschetz $\mathfrak{sl}(2)$-action, hence preserves the Riemann-Hodge form. ■

Sullivan's theorem

Theorem: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

Theorem: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\Big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}),q)$ **has finite kernel.**

Proof: Follows from the computation of $G = \operatorname{Aut}(H^*(M,\mathbb{R}), p_1, ..., p_n)$ done earlier. Indeed, the kernel of $\Gamma_0\Big|_{H^2(M,\mathbb{Z})}$ is a set of integer points of a compact Lie group, hence finite. The image of $\Gamma_0 = G_{\mathbb{Z}}$ has finite index in $O(H^2(M,\mathbb{Z}),q)$, because the corresponding map of Lie groups is surjective. ■

Computation of the mapping class group

COROLLARY: The mapping class group **Γ is mapped to $O(H^2(M,\mathbb{Z}),q)$ with finite kernel and finite index.**

Proof: By Sullivan, Γ is mapped to Γ_0 with finite kernel and finite index, and $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}),q)$ has finite kernel and finite index, as shown above. ■

THEOREM: (Kollar-Matsusaka, Huybrechts) **There are only finitely many connected components** of Teich.

COROLLARY: Let Γ_I be the group of elements of mapping class group preserving a connected component of Teichmüller space containing $I \in$ Teich. **Then Γ_I has finite index in Γ.**

REMARK: Γ_I is a group generated by monodromy of all Gauss-Manin local systems for all deformations of (M,I). It is known as **the monodromy group** of (M,I).

The period map

Remark: For any $J \in$ Teich, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let $P :$ Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map $P :$ Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

REMARK: P maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}\mathrm{er} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \mid \ q(l, l) = 0, q(l, \bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$

THEOREM: Let M be a simple hyperkähler manifold, and Teich its Teichmüller space. Then
(i) (Bogomolov) **The period map $P :$ Teich $\longrightarrow \mathbb{P}er$ is etale.**
(ii) (Huybrechts) It is **surjective.**

REMARK: Bogomolov's theorem implies that Teich **is smooth.** It is **usually non-Hausdorff**.

Birational equivalence and non-separable points

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (D. Huybrechts) If I_1, $I_2 \in$ Teich are non-separable points, then $P(I_1) = P(I_2)$, and (M, I_1) **is birationally equivalent** to (M, I_2).

COROLLARY: The set of all non-separable points in Teich **belongs to a union of countably many divisors.**

Proof. Step 1: Let $z \in H^2(M, \mathbb{Z})$ be a cohomology class, and Teich_z the set of all $I \in$ Teich such that $z \in H^{1,1}(M, I)$. The relation $z \in H^{1,1}(M, I)$ is equivalent to $q(z, \Omega) = 0$, hence

$$\operatorname{Per}(\operatorname{Teich}_z) \subset \{l \in \mathbb{P}(z^\perp) \ \mid \ q(l, l) = 0, q(l, \bar{l}) > 0.$$

This shows that $\operatorname{Per}(\operatorname{Teich}_z)$ **is a divisor.**

Step 2: If (M, I) is a holomorphic symplectic manifold with non-trivial birational equivalence to M', it contains a rational curve C. **Therefore,** $I \in \operatorname{Teich}_z$, **where** $z = [C]$ **is its homology class.** ■

Hausdorff reduction

REMARK: A non-Hausdorff manifold is a topological space locally diffeomorphic to $\mathbb{R}^n$.

DEFINITION: Let M be a topological space for which $M/\sim$ is Hausdorff. Then $M/\sim$ is called **a Hausdorff reduction** of M.

Problems:
1. **$\sim$ is not always an equivalence relation.**
2. **Even if $\sim$ is equivalence, the $M/\sim$ is not always Hausdorff.**

REMARK: A quotient $M/\sim$ is Hausdorff, if $M \longrightarrow M/\sim$ is open, and the graph $\Gamma_\sim \in M \times M$ is closed.

Weakly Hausdorff manifolds

DEFINITION: A point $x \in X$ is called **Hausdorff** if $x \not\sim y$ for any $y \neq x$.

DEFINITION: Let M be an n-dimensional real analytic manifold, not necessarily Hausdoff. Suppose that **the set $Z \subset M$ of non-Hausdorff points is contained in a countable union of real analytic subvarieties** of codim $\geqslant 2$. Suppose, moreover, that

(S) For every $x \in M$, there is a closed neighbourhood $B \subset M$ of x and a continuous surjective map $\Psi : B \longrightarrow \mathbb{R}^n$ to a closed ball in $\mathbb{R}^n$, **inducing a homeomorphism** on an open neighbourhood of x.

Then M is called **a weakly Hausdorff manifold**.

REMARK: The period map satisfies (S). Also, the non-Hausdorff points of Teich **are contained in a countable union of divisors.**

THEOREM: A **weakly Hausdorff manifold X admits a Hausdorff reduction.** In other words, the quotient $X/\sim$ is a Hausdorff. Moreover, $X \longrightarrow X/\sim$ is locally a homeomorphism.

This theorem is proven using 1920-ies style point-set topology.

Birational Teichmüller moduli space

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{\text{Per}} \mathbb{P}\text{er}$ **is an isomorphism,** for each connected component of Teich_b.

The proof is based on two results.

PROPOSITION: (The Covering Criterion) Let $X \xrightarrow{\varphi} Y$ be an etale map of smooth manifolds. Suppose that each $y \in Y$ has a neighbourhood $B \ni y$ diffeomorphic to a closed ball, such that for each connected component $B' \subset \varphi^{-1}(B)$, B' projects to B surjectively. **Then φ is a covering.**

PROPOSITION: The period map satisfies the conditions of the Covering Criterion.

Global Torelli theorem

DEFINITION: Let M be a hyperkaehler manifold, Teich_b its birational Teichmüller space, and Γ the mapping class group. The quotient $\operatorname{Teich}_b/\Gamma$ is called **the birational moduli space** of M.

REMARK: The birational moduli space is obtained from the usual moduli space **by gluing some (but not all) non-separable points. It is still non-Hausdorff.**

THEOREM: Let (M, I) be a hyperkähler manifold, and W a connected component of its birational moduli space. **Then W is isomorphic to $\mathbb{P}\mathrm{er}/\Gamma_I$, where $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ and Γ_I is an arithmetic subgroup in $O(H^2(M, \mathbb{R}), q)$.**

A CAUTION: Usually "the global Torelli theorem" is understood as a theorem about Hodge structures. For K3 surfaces, **the Hodge structure on $H^2(M, \mathbb{Z})$ determines the complex structure**. For $\dim_{\mathbb{C}} M > 2$, **it is false**.

The marked moduli space

DEFINITION: Let Γ be the mapping class group, and $K \subset \Gamma$ the kernel of the natural map $\Gamma \longrightarrow GL(H^2(M, \mathbb{Z}))$. **It is finite,** as we have shown. The quotient Teich/K is called **the marked moduli space**.

THEOREM: The natural map $\operatorname{Teich} \longrightarrow \operatorname{Teich}/K$ **is a homeomorphism on each connected component.**

Proof. Step 1:
Let $I \in \operatorname{Teich}$ be a fixed point of a subgroup $K_I \subset K$. By Bogomolov's theorem, $T_I \operatorname{Teich}$ is naturally identified with $H^{1,1}_I(M)$. Since the action of K on $H^2(M)$ is trivial, any $\alpha \in K_I$ acts trivially on $T_I \operatorname{Teich}$. Therefore, K_I **acts as identity on a connected component of** Teich **containing** I.

Step 2: From Step 1, obtain that **the quotient map** $\operatorname{Teich} \stackrel{\Psi}{\longrightarrow} \operatorname{Teich}/K$ **is a finite covering,** hence **it induces a finite covering of the corresponding Hausdorff reductions.** However, Ψ induces an isomorphism on each connected component of Teich_b, because **each component of** Teich_b **is isomorphic to** $\mathbb{P}\mathrm{er}$**.** ■

The Hodge-theoretic Torelli theorem

REMARK: The group $O(p,q)$ $(p,q > 0)$ has **4 connected components**, corresponding to the orientations of positive p-dimensional and negative q-dimensional planes.

DEFINITION: Let M be a hyperkaehler manifold. One says that **the Hodge-theoretic Torelli theorem holds for M** if

$$\mathrm{Teich}/\Gamma_I \longrightarrow \mathbb{P}\mathrm{er}/O^+(H^2(M,\mathbb{Z}),q),$$

where $O^+(H^2(M,\mathbb{Z}),q)$ is a subgroup of $O(H^2(M,\mathbb{Z}),q)$ preserving orientation on positive 3-planes. Equivalently, **it is true if M is uniquely determined by its Hodge structure**.

REMARK: The Hodge-theoretic Torelli theorem **is true for K3 surfaces**. **It is false** for all other known examples of hyperkaehler manifolds.

Problems:

1. The moduli space Teich/Γ **is not Hausdorff** (Debarre, 1984). Indeed, bimeromorphically equivalent hyperkähler manifolds have isomorphic Hodge structures.
2. **The covering** $\mathrm{Teich}_b/\Gamma_I \longrightarrow \mathbb{P}\mathrm{er}/O^+(H^2(M,\mathbb{Z}),q)$ **is non-trivial**, because the map $\Gamma_I \longrightarrow O^+(H^2(M,\mathbb{Z}),q)$ is not surjective (Namikawa, 2002).

The birational Hodge-theoretic Torelli theorem

DEFINITION: The birational Hodge-theoretic Torelli theorem is true for M if Γ_I (the stabilizer of a Torelli component in the mapping class group) **is isomorphic to** $O^+(H^2(M,\mathbb{Z}),q)$**.**

REMARK: If a birational Hodge-theoretic Torelli theorem holds for M, then any deformation of M is up to a bimeromorphic equivalence **determined by the Hodge structure on** $H^2(M)$**.**

THEOREM: (Markman) The for $M = K3^{[n]}$, the group Γ_I **is a subgroup of** $O^+(H^2(M,\mathbb{Z}),q)$ **generated by oriented reflections.**

THEOREM: Let $M = K3^{[n+1]}$ with n a prime power. Then the (usual) global Torelli theorem holds birationally: **two deformations of a Hilbert scheme with isomorphic Hodge structures are bimeromorphic**. **For other** n**, it is false** (Markman).