Subtwistor metric on the moduli of hyperkähler manifolds

Misha Verbitsky

Holomorphic symplectic varieties,
Courant Institute, New York University,

June 5, 2011.

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.**

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(b_2-3,3)$.** It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

The Teichmüller space and the mapping class group

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\widetilde{\mathsf{Teich}}$ the space of complex structures on M, and let $\mathsf{Teich} := \widetilde{\mathsf{Teich}}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **coarse moduli space of complex structures on M** is a connected component of Teich/Γ.

REMARK: For hyperkähler manifolds, we take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler**. It is open in the usual Teichmüller space.

The period map

Remark: For any $J \in \text{Teich}$, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let $P: \text{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map $P: \text{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

REMARK: P maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \mid \ q(l, l) = 0, q(l, \bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$

THEOREM: (Bogomolov) Let M be a simple hyperkähler manifold, and Teich its Teichmüller space. Then **The period map $P: \text{Teich} \longrightarrow \mathbb{P}er$ is etale.**

REMARK: Bogomolov's theorem implies that Teich **is smooth.** It is **non-Hausdorff** even in the simplest examples.

Hausdorff reduction

REMARK: A non-Hausdorff manifold is a topological space locally diffeomorphic to $\mathbb{R}^n$.

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (D. Huybrechts) If I_1, $I_2 \in \operatorname{Teich}$ are non-separablee points, then $P(I_1) = P(I_2)$, and (M, I_1) **is birationally equivalent** to (M, I_2)

DEFINITION: Let M be a topological space for which $M/\sim$ is Hausdorff. Then $M/\sim$ is called **a Hausdorff reduction** of M.

Problems:
1. **$\sim$ is not always an equivalence relation.**
2. **Even if $\sim$ is equivalence, the $M/\sim$ is not always Hausdorff.**

REMARK: A quotient $M/\sim$ is Hausdorff, if $M \longrightarrow M/\sim$ is open, and the graph $\Gamma_\sim \in M \times M$ is closed.

THEOREM: The Teichmüller space of a hyperkähler manifold **admits a Hausdorff reduction**.

Birational Teichmüller moduli space

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ is an isomorphism, for each connected component of Teich_b.

The proof is based on two results.

PROPOSITION: (The Covering Criterion) Let $X \xrightarrow{\varphi} Y$ be an etale map of smooth manifolds. Suppose that each $y \in Y$ has a neighbourhood $B \ni y$ diffeomorphic to a closed ball, such that for each connected component $B' \subset \varphi^{-1}(B)$, B' projects to B surjectively. **Then φ is a covering.**

PROPOSITION: The period map satisfies the conditions of the Covering Criterion.

Period space as a Grassmannian of positive 2-planes

PROPOSITION: The period space

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \ | \ \ q(l,l) = 0, q(l, \bar{l}) > 0.$$

is identified with $SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$, which is a Grassmannian of positive oriented 2-planes in $H^2(M,\mathbb{R})$.

Proof. Step 1: Given $l \in \mathbb{P}H^2(M,\mathbb{C})$, **the space generated by** $\operatorname{Im} l, \operatorname{Re} l$ **is 2-dimensional,** because $q(l,l) = 0, q(l,\bar{l})$ implies that $l \cap H^2(M,\mathbb{R}) = 0$.

Step 2: This 2-dimensional plane is positive, because $q(\operatorname{Re} l, \operatorname{Re} l) = q(l + \bar{l}, l + \bar{l}) = 2q(l, \bar{l}) > 0$.

Step 3: Conversely, for any 2-dimensional positive plane $V \in H^2(M,\mathbb{R})$, **the quadric** $\{l \in V \otimes_{\mathbb{R}} \mathbb{C} \ \ | \ \ q(l,l) = 0\}$ **consists of two lines;** a choice of a line is determined by orientation. ■

Period space and hyperkähler lines

DEFINITION: Let (M, I, J, K) be a hyperkähler manifold. **A hyperkähler 3-plane** in $H^2(M, \mathbb{R})$ is a positive oriented 3-dimensional subspace W, generated by $\omega_I, \omega_J, \omega_K$.

REMARK: The set of oriented 2-dimensional planes in W is identified with $S^2 = \mathbb{C}P^1$. It is called **the twistor family** of a hyperkähler structure. A point in the twistor family corresponds to a complex structure $aI + bJ + cK \in \mathbb{H}$, with $a^2 + b^2 + c^2 = 1$. We call the corresponding $\mathbb{C}P^1 \subset \operatorname{Teich}$ **the twistor curves**.

REMARK: Let $I \in \operatorname{Teich}$ be a complex structure, and $\mathcal{K}(I)$ its Kähler cone. The set of twistor curves passing through I **is parametrized by $\mathcal{K}(I)$,** by Calabi-Yau theorem. **The corresponding 3-dimensional subspaces are generated by $\operatorname{Per}(I) + \omega$, where $\omega \in \mathcal{K}(I)$.**

Divisorial Zariski decomposition

DEFINITION: A class $\eta \in H^{1,1}(M)$ is called **pseudoeffective** if it can be represented by a positive current, and **nef** if it lies in the closure of a Kaehler cone.

DEFINITION: A **modified nef cone** (also "birational nef cone" and "movable nef cone") is a closure of a union of all nef cones for all bimeromorphic models of a holomorphically symplectic manifold M.

THEOREM: (D. Huybrechts, S. Boucksom)
The modified nef cone is dual to the pseudoeffective cone under the Bogomolov-Beauville-Fujiki pairing.

The divisorial Zariski decomposition theorem: (S. Boucksom)
Let M be a simple hyperkähler manifold. Then **every pseudoeffective class can be decomposed as a sum** $\eta = \nu + \sum_i a_i[E_i]$, where ν is modified nef, a_i positive numbers, and E_i exceptional divisors satisfying $q(E_i, E_i) < 0$.

COROLLARY: Let M be a hyperkaehler manifold with $H^{1,1}(M, \mathbb{Z}) = 0$. **Then every pseudoeffective class is modified nef.**

Proof: Indeed, on such M **there are no exceptional divisors.** ■

A Kähler cone for generic hyperkähler manifolds

COROLLARY: Let M be a hyperkaehler manifold with

$$\mathsf{NS}(M) := H^{1,1}(M, \mathbb{Z}) = 0.$$

Then the modified nef cone is self-dual.

REMARK: For any nef classes ν, ν', one has $q(\nu, \nu') \geqslant 0$. Therefore, **the nef cone is contained in its dual.** Moreover, the nef cone K_n is contained in one of two components K^+ of a set $\{\nu \in H^{1,1}(M) \mid q(\nu, \nu) \geqslant 0\}$, and therefore, **the dual nef cone contains K^+:**

$$K_n \subset K^+ \subset K_n^*$$

REMARK: For $\mathsf{NS}(M) = 0$, the modified nef cone K_{mn} is self-dual, but all elements of K_{mn} satisfy $q(\nu, \nu) \geqslant 0$. This gives

$$K_{mn} \subset K^+ \subset K_{mn}^* = K_{mn} \subset K^+.$$

We obtain

COROLLARY: Let M be a hyperkaehler manifold with $\mathsf{NS}(M) = 0$. **Then the Kaehler cone is one of two components of the set**

$$\{\nu \in H^{1,1}(M) \mid q(\nu, \nu) \geqslant 0\}.$$

■

Generic hyperkähler lines

DEFINITION: Let $S \subset \text{Teich}$ be a $\mathbb{C}P^1$ associated with a twistor family. It is called **generic** if it passes through a point $I \in \text{Teich}$ with $\text{NS}(M,I) = 0$.

REMARK: For a generic point I in such S, one has $\text{NS}(M,I) = 0$. **This condition is equivalent to $l^\perp \cap H^2(M,\mathbb{Z}) = 0$,** where $l \in \mathbb{P}\text{er}$ is the corresponding 2-plane.

REMARK: A 3-plane $W \subset H^2(M,\mathbb{R})$ corresponds to a generic twistor family if and only if its orthogonal complement $W^\perp \subset H^2(M,\mathbb{R})$ does not contain rational vectors.

DEFINITION: A hyperkähler 3-plane $W \subset H^2(M,\mathbb{R})$ is called **generic** if $W^\perp \cap H^2(M,\mathbb{Z}) = 0$. The corresponding $\mathbb{C}P^1 \subset \mathbb{P}\text{er}$ in the period space is called **a GHK line**.

A lifting property for GHK lines

REMARK: Consider a 3-plane $W = \langle \omega_I, \omega_J, \omega_K \rangle$ associated with a hyperkähler structure, and let S be the set of oriented 2-planes in W. Denote by S_{ng} the set of $x \in S$ satisfying $x^\perp \cap H^2(M, \mathbb{Z}) \neq 0$. If W is generic, then S_{ng} is countable.

THEOREM: (A lifting property for GHK lines)
Let $W \subset H^2(M, \mathbb{R})$ be a generic 3-plane, and $S \subset \mathbb{P}\mathrm{er}$ the corresponding GHK line. Consider the period map $P : \operatorname{Teich} \longrightarrow \mathbb{P}\mathrm{er}$. **Then $P^{-1}(S)$ is a union of a countable set mapped to S_{ng}, and a disconnected set of rational curves bijectively mapped to S.**

Proof. Step 1: Let $x \notin S_{ng}$ We are going to prove that for all $I \in P^{-1}(x)$, **y is contained in a connected component of $P^{-1}(S)$, bijectively mapped to S.**

Step 2: Notice that $NS(I) = x^\perp \cap H^2(M, \mathbb{Z}) = 0$. Therefore the Kähler cone of (M, I) **is one of two components of the set $\{\omega \in P(I)^\perp \mid q(\omega, \omega) > 0\}$.**.

Step 3: For each positive 3-plane $W \subset H^2(M, \mathbb{R})$, $W = \langle \omega_I, \omega_J, \omega_K \rangle$ for some hyperkähler structure I, J, K. **Then the twistor family associated with I, J, K is mapped to S.** ■

The covering condition and the lifting property

We recall the covering criterion stated above.

PROPOSITION: (The Covering Criterion) Let $X \xrightarrow{\varphi} Y$ be an etale map of smooth manifolds. Suppose that each $y \in Y$ has a neighbourhood $B \ni y$ diffeomorphic to a closed ball, such that for each connected component $B' \subset \varphi^{-1}(B)$, B' projects to B surjectively. **Then φ is a covering.**

THEOREM: (*) Let $P: X \longrightarrow \mathbb{P}\mathrm{er}$ be an etale map which satisfies the GHK lifting property. **Then P satisfies the assumptions of the Covering Criterion.**

COROLLARY: This result **implies that the period map $\mathrm{Teich}_b \xrightarrow{P} \mathbb{P}\mathrm{er}$ is a diffeomoirphism.**

To prove Theorem (*), we introduce **the subtwistor metric** on Teich.

DEFINITION: Let g_0 be a Riemannian metric on $\mathbb{P}\mathrm{er}$, and g its lift to Teich. Define **the subtwistor metric** d as the distance function $d(x,y)$ given by infimum of the length (in g) for all paths from x to y going through GHK curves which intersect in points $z \in \mathrm{Teich}$ with $\mathrm{NS}(z) = 0$.

Subtwistor metric on the Teichmüller space

CLAIM: The subtwistor metric induces the standard topology on any open subset $W \subset \mathrm{Teich}_b$.

Remark: Its proof follows from Gleason-Palais-Montgomery classification of continuous groups.

THEOREM: Let $W_o \in \mathbb{P}\mathrm{er}$ be an open, connected subset, with smooth boundary, and W its closure. Consider a connected component $W_1 \subset \mathrm{Teich}_b$ of $\mathrm{Per}^{-1}(W)$. **Then $\mathrm{Per} : W_1 \longrightarrow W$ is surjective.**

Proof. Step 1: Whenever $x, y \in W$ are connected by segments of GHK curves which lie in W, and $x \in W_1$, one has $y \in W_1$, by lifting property of GHK curves. Therefore, **it would suffice to prove that all W is connected by segments of GHK curves** which lie in W.

Step 2: This is the same as to say that W is connected in the topology induced by subtwistor metric. By the previous claim, **this is equivalent to connectedness of W.** ■