

Symplectic packing and hyperbolicity

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Motivation: dominating maps to hyperkähler manifolds

DEFINITION: A complex n -manifold M is **dominated by \mathbb{C}^n** if there exists a holomorphic map $\mathbb{C}^n \rightarrow M$ with non-zero Jacobian at some point.

THEOREM: (Buzzard, Lu)

Kummer K3 surfaces are dominated by \mathbb{C}^2 .

REMARK: We know now that the hyperkähler manifolds admit many entire curves.

QUESTION: Are all $2n$ -dimensional holomorphic symplectic manifolds dominated by \mathbb{C}^{2n} ?

REMARK: This is unknown even for a K3.

Motivation: holomorphic symplectic packing

The following theorem easily follows from ergodicity.

DEFINITION: A **holomorphic symplectic ball** $B_r^{\mathbb{C}}$ of radius r is an open ball in \mathbb{C}^{2n} of radius r with standard symplectic form.

THEOREM: Let M be a compact, holomorphically symplectic manifold of Kähler type, $\dim_{\mathbb{C}} M = 2n$, $I(M)$ the supremum of r for which there exists a holomorphically symplectic immersion $B_r^{\mathbb{C}} \rightarrow M$, and $E(M)$ the supremum of r for which there exists a holomorphically symplectic embedding $B_r^{\mathbb{C}} \rightarrow M$.

Then $E(M)$ and $I(M)$ stay constant under deformations of M , as long as $\text{Pic}(M)$ is not maximal and $\text{Vol}(M)$ stays constant.

CONJECTURE: If $I(M) = \infty$, then M is dominated by \mathbb{C}^{2n} .

REMARK: So far there is no way to compute holomorphic symplectic invariants $I(M)$ and $E(M)$, but **similar constants from symplectic geometry are possible to compute.**

Full symplectic packing

DEFINITION: A **symplectic ball** is a ball of radius r in \mathbb{R}^{2n} , equipped with a standard symplectic structure $\omega = \sum dp_i \wedge dq_i$.

DEFINITION: Let M be a compact symplectic manifold of volume V . We say that M **admits a full symplectic packing** if for any disconnected union S of symplectic balls of total volume less than V , S admits a symplectic embedding to M .

DEFINITION: A symplectic structure ω on a torus is called **standard** if there exists a flat torsion-free connection preserving ω .

THEOREM: (Latschev, McDuff, Schlenk, 2011)

All 4-dimensional tori with standard symplectic structures admit full symplectic packing.

Main result

DEFINITION: Let M be a compact symplectic manifold of volume V . We say that M **admits a full symplectic packing** if for any disconnected union S of symplectic balls of total volume less than V , S admits a symplectic embedding to M .

DEFINITION: A symplectic structure ω on a torus is called **standard** if there exists a flat torsion-free connection preserving ω . A symplectic structure ω on a hyperkähler manifold is called **standard** if ω is a Kähler form for some hyperkähler structure.

REMARK: Any known symplectic structure on a hyperkähler manifold or a torus is of this type. **It was conjectured that non-standard symplectic structures don't exist.**

THEOREM: (Entov-V.)

Let M be a compact even-dimensional torus, or a hyperkähler manifold (such as a K3 surface), and ω a standard symplectic form. **Then (M, ω) admits a full symplectic packing.**

REMARK: In this talk, **all tori are compact, even-dimensional, and satisfy $\dim_{\mathbb{R}} M \geq 4$.**

Motivation: Gromov Capacity

DEFINITION: Let M be a symplectic manifold. Define **Gromov capacity** $\mu(M)$ as the supremum of radii r , for all symplectic embeddings from a symplectic ball B_r to M .

DEFINITION: Define **symplectic volume** of a symplectic manifold (M, ω) as $\int_M \omega^{\frac{1}{2} \dim_{\mathbb{R}} M}$.

REMARK: Gromov capacity is obviously bounded by the symplectic volumes: a manifold of Gromov capacity r has volume $\geq \text{Vol}(B_r)$. However, **there are manifolds of infinite volume with finite Gromov capacity.**

THEOREM: (Gromov)

Consider **a symplectic cylinder** $C_r := \mathbb{R}^{2n-2} \times B_r$ with the product symplectic structure. Then the Gromov capacity of C_r is r .

REMARK: This result was used by Gromov to study symplectic packing in $\mathbb{C}P^2$. He proved that **there is no full symplectic packing**, and found precise bounds.

Ekeland-Hofer theorem

THEOREM: (Ekeland-Hofer)

Let M, N be symplectic manifolds, and $\varphi : M \rightarrow N$ a diffeomorphism. Suppose that for all sufficiently small, convex open sets $U \subset M$, Gromov capacity satisfies $\mu(U) = \mu(\varphi(U))$. **Then φ is a symplectomorphism.**

REMARK: This can be used to define C^0 - (continuous) symplectomorphisms.

REMARK: Ekeland-Hofer theorem implies a theorem of Gromov-Eliashberg: **symplectomorphism group is C^0 -closed in the group of diffeomorphisms.**

McDuff and Polterovich for Kähler manifolds

DEFINITION: Let M be a symplectic manifold, $x_1, \dots, x_n \in M$ distinct points, and r_1, \dots, r_n a set of positive numbers. We say that M **admits symplectic packing** with centers x_1, \dots, x_n and radii r_1, \dots, r_n if there exists a symplectic embedding from a disconnected union of symplectic balls of radii r_1, \dots, r_n to M mapping centers of balls to x_1, \dots, x_n .

THEOREM: (McDuff, Polterovich, 1995)

Let (M, ω) be a Kähler manifold, $\tilde{M} \xrightarrow{\pi} M$ its blow-up in x_1, \dots, x_n , E_i the corresponding exceptional divisors, and $[E_i]$ their fundamental classes. Assume that the class $\pi^*\omega - \sum_i c_i [E_i]$ is Kähler, for some $c_i > 0$. **Then M admits a symplectic packing with radii $r_i = \pi^{-1} \sqrt{c_i}$.**

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K , satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K .

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K .

COROLLARY: The group $SU(2)$ of orthogonal quaternions acts on triples (I, J, K) producing new hyperkähler structures.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M .

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Calabi-Yau and Bogomolov decomposition theorem

REMARK: A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1}\omega_K$ is a holomorphic symplectic form on (M, I) .

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

CLAIM: A compact hyperkähler manifold M **has maximal holonomy of Levi-Civita connection $Sp(n)$** if and only if $\pi_1(M) = 0$, $h^{2,0}(M) = 1$.

THEOREM: (Bogomolov decomposition)

Any compact hyperkähler manifold has a finite covering isometric to a product of a torus and several maximal holonomy hyperkähler manifolds.

Campana simple manifolds

DEFINITION: A complex manifold M , $\dim_{\mathbb{C}} M > 1$, is called **Campana simple** if the union \mathcal{U} of all complex subvarieties $Z \subset M$ satisfying $0 < \dim Z < \dim M$ has measure 0. A point which belongs to $M \setminus \mathcal{U}$ is called **generic**.

REMARK: Campana simple manifolds are non-algebraic. Indeed, a manifold which admits a globally defined meromorphic function f is a union of zero divisors for the functions $f - a$, for all $a \in \mathbb{C}$, and the zero divisor for f^{-1} . Hence **Campana simple manifolds admit no globally defined meromorphic functions.**

EXAMPLE: A general complex torus has no non-trivial complex subvarieties, hence it is Campana simple.

EXAMPLE: Let (M, I, J, K) be a hyperkähler manifold, and $L = aI + bJ + cK$, $a^2 + b^2 + c^2 = 1$ be a complex structure induced by quaternions. Then for all such (a, b, c) outside of a countable set, **all complex subvarieties $Z \subset (M, L)$ are hyperkähler, and (unless M a finite quotient of a product) $\cup_Z Z \neq M$** (V., 1994, 1996). **Therefore, (M, L) is Campana simple.**

CONJECTURE: (Campana)

Let M be a Campana simple Kähler manifold. **Then M is bimeromorphic to a finite quotient of a hyperkähler orbifold or a torus.**

Demailly-Paun theorem

REMARK: Let M be a compact Kähler manifold. Recall that the cohomology space $H^2(M, \mathbb{C})$ is decomposed as $H^2(M, \mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$ with $H^{1,1}(M)$ identified with the space of I -invariant harmonic 2-forms, and $H^{2,0}(M) \oplus H^{0,2}(M)$ the space of I -antiinvariant harmonic 2-forms. This decomposition is called **Hodge decomposition**. The space $H^{1,1}(M)$ is a complexification of a real space $H^{1,1}(M, \mathbb{R}) = \{\nu \in H^2(M, \mathbb{R}) \mid I(\nu) = \nu\}$.

THEOREM: (Demailly-Păun, 2002)

Let M be a compact Kähler manifold, and $\hat{K}(M) \subset H^{1,1}(M, \mathbb{R})$ a subset consisting of all $(1,1)$ -forms η which satisfy $\int_Z \eta^k > 0$ for any k -dimensional complex subvariety $Z \subset M$. **Then the Kähler cone of M is one of the connected components of $\hat{K}(M)$.** ■

Kähler cone for blow-ups of Campana simple manifolds

Theorem 2: Let M be a Campana simple compact Kähler manifold, and x_1, \dots, x_n distinct generic points of M . Consider the blow-up \tilde{M} of M in x_1, \dots, x_n , let E_i be the corresponding blow-up divisors, and $[E_i] \in H^2(M, \mathbb{Z})$ its fundamental classes. Decompose $H^{1,1}(\tilde{M}, \mathbb{R})$ as $H^{1,1}(\tilde{M}, \mathbb{R}) = H^{1,1}(M, \mathbb{R}) \oplus \bigoplus \mathbb{R}[E_i]$. Assume that η_0 is a Kähler class on M . **Then for any $\eta = \eta_0 + c_i[E_i]$, the following conditions are equivalent.**

- (i) η is Kähler on \tilde{M} .
- (ii) all c_i are negative, and $\int_M \eta^{\dim_{\mathbb{C}} M} > 0$.

Proof of (ii) \Rightarrow (i). Step 1:

All proper complex subvarieties of \tilde{M} are either contained in E_i , or do not intersect E_i . The condition “ η_0 is Kähler on M ” implies $\int_Z \eta^k > 0$ for all subvarieties not intersecting E_i . Since $[E_i]$ restricted to E_i is $-\omega_{E_i}$, where ω_{E_i} is the Fubini-Study form, $c_i < 0$ implies that $\int_Z \eta^k > 0$ for all subvarieties which lie in E_i . Finally, the integral of η over \tilde{M} is positive by the assumption $\int_M \eta^{\dim_{\mathbb{C}} M} > 0$. **Therefore, the condition (ii) implies that $\eta \in \hat{K}(\tilde{M})$.**

Kähler cone for blow-ups of Campana simple manifolds (cont.)

Theorem 2: Let M be a Campana simple compact Kähler manifold, and x_1, \dots, x_n distinct generic points of M . Consider the blow-up \tilde{M} of M in x_1, \dots, x_n , let E_i be the corresponding blow-up divisors, and $[E_i] \in H^2(M, \mathbb{Z})$ its fundamental classes. Decompose $H^{1,1}(\tilde{M}, \mathbb{R})$ as $H^{1,1}(\tilde{M}, \mathbb{R}) = H^{1,1}(M, \mathbb{R}) \oplus \bigoplus \mathbb{R}[E_i]$. Assume that η_0 is a Kähler class on M . **Then for any $\eta = \eta_0 + c_i[E_i]$, the following conditions are equivalent.**

- (i) η is Kähler on \tilde{M} .
- (ii) all c_i are negative, and $\int_M \eta^{\dim_{\mathbb{C}} M} > 0$.

Proof of (ii) \Rightarrow (i). Step 2:

The form η_0 is Kähler on M , hence it lies on the boundary of the Kähler cone of \tilde{M} , and η_0 can be obtained as a limit

$$\eta_0 = \lim_{\varepsilon \rightarrow 0} \eta_0 + \varepsilon c_i [E_i]$$

of forms which lie in the same connected component of $\hat{K}(\tilde{M})$. Therefore, η **belongs to the same connected component of $\hat{K}(\tilde{M})$ as a Kähler form.** By Demailly-Păun, this implies that η is Kähler.

Proof of (i) \Rightarrow (ii).

The numerical conditions of (ii) mean that $\eta \in \hat{K}(\tilde{M})$, hence they are satisfied automatically, as follows from Step 1. ■

Campana simple manifolds and symplectic packings

DEFINITION: Let M be a compact symplectic manifold of volume V . We say that M **admits a full symplectic packing** if for any disconnected union S of symplectic balls of total volume less than V , S admits a symplectic embedding to M .

Theorem 3: Let (M, I, ω_0) be a Kähler, compact, Campana simple manifold. **Then M admits a full symplectic packing.**

Proof. Step 1: Let x_1, \dots, x_n distinct generic points of M . Consider the blow-up \tilde{M} of M in x_1, \dots, x_n , let E_i be the corresponding blow-up divisors, and $[E_i] \in H^2(M, \mathbb{Z})$ their fundamental classes. As follows from McDuff-Polterovich, existence of full symplectic packing on M is implied by existence of a Kähler form $\omega(c_1, \dots, c_n)$ on \tilde{M} with cohomology class $[\omega(c_1, \dots, c_n)] = [\omega_0] - \sum c_i [E_i]$ for all (c_1, \dots, c_n) satisfying $\int_{\tilde{M}} ([\omega_0] - \sum c_i [E_i])^n > 0$

Step 2: Such a form exists by Theorem 2. ■

Symplectic packing on hyperkähler manifolds and compact tori with irrational symplectic form

DEFINITION: A symplectic form is called **irrational** if its cohomology class is irrational, that is, lies in $H^2(M, \mathbb{R}) \setminus \mathbb{R} \cdot H^2(M, \mathbb{Q})$.

THEOREM: Let M be a hyperkähler manifold or a compact torus, ω an irrational, standard symplectic form, and \mathcal{T} the set of complex structures for which ω is Kähler. **Then the set $\mathcal{T}_0 \subset \mathcal{T}$ of Campana simple complex structures is dense in \mathcal{T} and has full measure in the corresponding moduli space.**

Proof: The Hodge loci of hyperkähler manifolds admitting a non-hyperkähler subvariety have positive codimension, and deformations of hyperkähler subvarieties never cover M . ■

COROLLARY: Let M be a hyperkähler manifold or a compact torus, equipped with a standard, irrational symplectic form ω . **Then M admits full symplectic packing.**

Proof: By definition of a standard symplectic form, there exists a complex structure I such that ω is Kähler. Deforming I in \mathcal{T} , we obtain a Campana simple complex structure for which ω is Kähler. Then (M, ω) admits full symplectic packing by Theorem 3. ■

Symplectic cone and Kähler cone

DEFINITION: An almost complex structure I **tames** a symplectic structure ω if $\omega_I^{1,1}$ is a Hermitian form on (M, I) .

PROPOSITION: Let (M, I, ω) be an almost complex tamed symplectic manifold, and $\eta \in \Lambda^{2,0+0,2}(M, \mathbb{R})$ a closed real $(2, 0) + (0, 2)$ -form. **Then $\omega + \eta$ is also a symplectic form.**

Proof. Step 1: Since $I(\eta) = -\eta$ for each $(2, 0) + (0, 2)$ -form η , one has $\omega(x, Ix) = \omega^{1,1}(x, Ix) > 0$ for each non-zero x .

Step 2: Since $\eta^{1,1} = 0$, one has $(\omega + \eta)(x, Ix) = \omega^{1,1}(x, Ix) > 0$ for each non-zero x . Therefore, $\omega + \eta$ is non-degenerate. ■

DEFINITION: A **symplectic class** of a manifold M is a cohomology class of a symplectic form on M . **Symplectic cone** of a symplectic manifold M is a set $\text{Symp}(M) \subset H^2(M, \mathbb{R})$ of all symplectic classes.

Corollary 1: Let M be a Kähler manifold, and $\text{Kah}(M)$ its Kähler cone. **Then the symplectic cone of M contains $\text{Kah}(M) + H^{2,0+0,2}(M, \mathbb{R})$.** ■

Symplectic cone for blown-up tori and hyperkähler manifolds

Theorem 4: Let (M, I, ω_I) be a compact Kähler manifold obtained as a limit of Campana simple manifolds. Then (M, ω_I) admits full symplectic packing.

Proof. Step 1: Let B be an open neighbourhood of I in the moduli space of complex structures on M , and $B \xrightarrow{\varphi} H^2(M, \mathbb{R})$ a map putting J to $(\omega_I)_J^{1,1}$. By Kodaira stability theorem, $\varphi(J)$ is a Kähler class for J sufficiently close to I . Therefore, **there exists a Campana simple complex structure J such that $\omega_J := (\omega_I)_J^{1,1}$ is Kähler, arbitrarily close to I in B .**

Step 2: By Theorem 1, $\eta_J := \omega_J + \sum c_i [E_i]$ is a Kähler class on a blow-up of (M, J) , with blow-up points generic. Indeed, the condition $\int_M \eta_J^{\dim_{\mathbb{C}} M} > 0$ remains true for J sufficiently close to I .

Step 3: Now, $\omega_I - \omega_J$ is by definition a $(2, 0) + (0, 2)$ -cohomology class on (M, J) . Therefore, $\eta = \omega + \sum c_i [E_i]$ is obtained from a Kähler form η_J by adding a $(2, 0) + (0, 2)$ -form on (M, J) . ■

This implies Theorem 1.

Further directions

1. We explored symplectic packing by symplectic balls. What about a packing by other subsets $K \subset \mathbb{R}^{2n}$?

1A. Define a packing number $\nu(K, M)$ of (K, ω) to M as a supremum of all ε for which $(K, \varepsilon\omega)$ admits a symplectic embedding to M . This function is obviously semicontinuous on K and M . When K is a union of symplectic balls, and M a hyperkähler manifold or a torus, $\nu(K, M) = \frac{\text{Vol}(M)}{\text{Vol}(K)}$. Using ergodicity, it is possible to show that $\frac{\nu(K, M)}{\text{Vol}(M)}$ is constant for irrational symplectic structures on such M . Is it equal to 1? If so, we have “full packing by K ”.

2. Replacing blow-ups by orbifold blow-ups and balls by symplectic ellipsoids with rational axis length, our argument would give full packing by ellipsoids.

Further directions (cont.)

3. Let Symp be the infinite-dimensional Frechet manifold of all symplectic forms on M , and Diff the diffeomorphism group. The full packing phenomena seems to be related to ergodicity of Diff -action on Symp : the packing defines a semi-continuous, Diff -invariant function on Symp , which should be a posteriori constant on the set of all symplectic structures with dense Diff -orbits. One could study other semi-continuous quantities in relation to Diff -action and ergodicity.

4. Let Diff_0 be the group of isotopies (connected component of Diff). The symplectic Teichmüller space $\text{Teich}_s := \text{Symp} / \text{Diff}_0$ is locally diffeomorphic to $H^2(M, \mathbb{R})$ by Moser's theorem. The arguments similar to given above would show that standard connected components of Teich_s are embedded to $H^2(M, \mathbb{R})$ (for hyperkähler manifolds), and the image is a defined by a single quadratic inequality. What about other symplectic manifolds? Is Teich_s Hausdorff? Embedded to $H^2(M, \mathbb{R})$?

APPENDIX: WON'T BE SHOWN

unless demanded

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I : TM \rightarrow TM$ which satisfies $I^2 = -\text{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)

This definition is equivalent to the usual one.

Kähler manifolds

DEFINITION: A Riemannian metric g on an almost complex manifold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g) .

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

- (i) The complex structure I is integrable, and the Hermitian form ω is closed.
- (ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla : \text{End}(TM) \longrightarrow \text{End}(TM) \otimes \Lambda^1(M).$$

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M . The set of all Kähler classes is called **the Kähler cone**.

Kähler structure on a blow-up

DEFINITION: Let S be a total space of a line bundle $\mathcal{O}(-1)$ on $\mathbb{C}P^n$, identified with a space of pairs $(z \in \mathbb{C}P^n, t \in z)$, where t is a point on a line $z \subset \mathbb{C}^{n+1}$ representing z . The forgetful map $\pi : S \rightarrow \mathbb{C}^{n+1}$ is called **a blow-up of \mathbb{C}^{n+1} in 0** . Given an open ball $B \subset \mathbb{C}^{n+1}$, the map $\pi : \pi^{-1}(B) \rightarrow B$ is called **a blow-up of B in 0** . To blow up a point in a complex manifold M , we remove a ball B around this point, and replace it with a blown-up ball \tilde{B} , gluing $B \setminus x \subset \tilde{B}$ with $B \setminus x \subset M$.

PROBLEM: Suppose that M is Kähler, and \tilde{M} is its blow-up. **Find a Kähler metric on \tilde{M} and write it explicitly.**

Answer: Symplectic blow-up!

REMARK: In this talk, I would often drop all π and other constants from the equations.

Symplectic quotient

DEFINITION: Let ρ be an S^1 -action on a symplectic manifold (M, ω) preserving the symplectic structure, and \vec{v} its unit tangent vector. Cartan's formula gives $0 = \text{Lie}_{\vec{v}}\omega = d(\omega \lrcorner \vec{v})$, hence $\omega \lrcorner \vec{v}$ is a closed 1-form. **Hamiltonian**, or **moment map** of ρ is an S^1 -invariant function μ such that $d\mu = \omega \lrcorner \vec{v}$, and **symplectic quotient** $M //_c S^1$ is $\mu^{-1}(c)/S^1$.

REMARK: In these assumptions, restriction of the symplectic form ω to $\mu^{-1}(c)$ vanishes on \vec{v} , hence it is **obtained as a pullback of a closed 2-form $\omega_{//}$ on $M //_c S^1$** .

THEOREM: **The form $\omega_{//}$ is a symplectic form on $M //_c S^1$** . In other words, **the symplectic quotient is a symplectic manifold**.

REMARK: If, in addition, M is equipped with a Kähler structure (I, ω) , and S^1 -action preserves the complex structure, the symplectic quotient $M //_c S^1$ inherits the Kähler structure. In this case it is called **a Kähler quotient**. Whenever the S^1 -action can be integrated to holomorphic \mathbb{C}^* -action, **the Kähler quotient is identified with an open subset of its orbit space**.

REMARK: The moment map is defined by $d\mu = \omega \lrcorner \vec{v}$ uniquely up to a constant. However, the symplectic quotient $M //_c S^1 = \mu^{-1}(c)/S^1$ **depends heavily on the choice of $c \in \mathbb{R}$** .

Symplectic blow-up

CLAIM: Consider the standard S^1 -action on \mathbb{C}^n , and let $W \subset \mathbb{C}^n$ be an S^1 -invariant open subset. Consider the product $V := W \times \mathbb{C}$ with the standard symplectic structure and take the S^1 -action on \mathbb{C} opposite to the standard one. **Then its moment map is $w - t$, where $w(x) = |x|^2$ is the length function on W and $r(t) = |t|^2$ the length function on \mathbb{C} .**

DEFINITION: Symplectic cut of W is $(W \times \mathbb{C}) //_c S^1$.

REMARK: Geometrically, the symplectic cut is obtained as follows. Take $c \in \mathbb{R}$, and let $W_c := \{w \in W \mid |w|^2 \leq c\}$. Then W_c is a manifold with boundary ∂W_c , which is a sphere $|w|^2 = c$. Then $(W \times \mathbb{C}) //_c S^1 = (W_c \times \mathbb{C}) //_c S^1$ is obtained from W_c by gluing each S^1 -orbit which lies on ∂W_c to a point. Combinatorially, $(W \times \mathbb{C}) //_c S^1$ is \mathbb{C}^n with 0 replaced with $\mathbb{C}P^{n-1}$.

DEFINITION: In these assumptions, **symplectic blow-up** of radius $\lambda = \sqrt{c}$ of W in 0 is $(W \times \mathbb{C}) //_c S^1$. **Symplectic blow-up** of a symplectic manifold M is obtained by removing a symplectic ball W and gluing back a blown-up symplectic ball $(W \times \mathbb{C}) //_c S^1$.

REMARK: The **symplectic form ω_c on the blow-up $(W \times \mathbb{C}) //_c S^1$ depends on c as follows:** $\int_l \omega_c = c$, where $l \subset E$ is a rational line on an exceptional divisor $E := \pi^{-1}(c)$.

McDuff and Polterovich: symplectic packing from symplectic blow-ups

DEFINITION: Let M be a symplectic manifold, $x_1, \dots, x_n \in M$ distinct points, and r_1, \dots, r_n a set of positive numbers. We say that M **admits symplectic packing** with centers x_1, \dots, x_n and radii r_1, \dots, r_n if there exists a symplectic embedding from a disconnected union of symplectic balls of radii r_1, \dots, r_n to M mapping centers of balls to x_1, \dots, x_n .

REMARK: The choice of x_i is irrelevant, because **the group of symplectic automorphisms acts on M infinitely transitively**.

Theorem 1: (McDuff-Polterovich)

Let (M, ω) be a symplectic manifold, $x_1, \dots, x_n \in M$ distinct points, and c_1, \dots, c_n a set of positive numbers. Let $\pi : \tilde{M} \rightarrow M$ be a symplectic blow-up with centers in x_i , and $E_i \in H^2(\tilde{M}, \mathbb{Z})$ the fundamental classes of its exceptional divisors. Then the following conditions are equivalent.

- (i) **M admits a symplectic packing with radii $r_i = \pi^{-1} \sqrt{c_i}$**
- (ii) For any $\varepsilon \in [0, 1]$, **there exists a form $\omega_\varepsilon(c_1, \dots, c_n)$ cohomologically equivalent to $\pi^* \omega - \sum \varepsilon \pi^* c_i E_i$** , symplectic for $\varepsilon > 0$, smoothly depending on ε , and satisfying $\omega_0(c_1, \dots, c_n) = \pi^* \omega$. ■

McDuff and Polterovich for Kähler manifolds

REMARK: In Kähler situation, the smooth dependence condition is **trivial**, because for any two Kähler forms ω, ω' , straight interval connecting ω to ω' consists of Kähler forms (indeed, **the set of Kähler forms is convex**). This brings the following corollary.

Corollary 1: Let (M, ω) be a Kähler manifold, $\tilde{M} \xrightarrow{\pi} M$ its blow-up in x_1, \dots, x_n , E_i the corresponding exceptional divisors, and $[E_i]$ their fundamental classes. Assume that the class $\pi^*\omega - \sum_i c_i [E_i]$ is Kähler, for some $c_i > 0$. **Then M admits a symplectic packing with radii $r_i = \pi^{-1} \sqrt{c_i}$.**