Symplectic packing and hyperkähler geometry

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Joint work with Misha Entov

Motivation: dominating maps to hyperkähler manifolds

DEFINITION: A complex n-manifold M is **dominated by** $\mathbb{C}^n$ if there exists a holomorphic map $\mathbb{C}^n \longrightarrow M$ with non-zero Jacobian at some point.

THEOREM: (Buzzard, Lu)
Kummer K3 surfaces are dominated by $\mathbb{C}^2$.

REMARK: We know now that the hyperkähler manifolds admit many entire curves.

QUESTION: Are all $2n$-dimensional holomorphic symplectic manifolds dominated by $\mathbb{C}^{2n}$?

REMARK: This is unknown even for a K3.

QUESTION: What is the largesr r such that a holomorphic symplectic ball of radius r can be embedded to a given compact holomorphically symplectic manifold of volume V?

REMARK: This number is essentially deformational invariant, by ergodicity.

Full symplectic packing

DEFINITION: A **symplectic ball** is a ball of radius r in $\mathbb{R}^{2n}$, equipped with a standard symplectic structure $\omega = \sum dp_i \wedge dq_i$.

DEFINITION: Let M be a compact symplectic manifold of volume V. We say that M **admits a full symplectic packing** if for any disconnected union S of symplectic balls of total volume less than V, S admits a symplectic embedding to M.

DEFINITION: A symplectic structure ω on a torus is called **standard** if there exists a flat torsion-free connection preserving ω.

THEOREM: (Latschev, McDuff, Schlenk, 2011)
All 4-dimensional tori with standard symplecic structures admit full symplectic packing.

Main result

DEFINITION: Let M be a compact symplectic manifold of volume V. We say that M **admits a full symplectic packing** if for any disconnected union S of symplectic balls of total volume less than V, S admits a symplectic embedding to M.

DEFINITION: A symplectic structure ω on a torus is called **standard** if there exists a flat torsion-free connection preserving ω. A symplectic structure ω on a hyperkähler manifold is called **standard** if ω is a Kähler form for some hyperkähler structure.

REMARK: Any known symplectic structure on a hyperkähler manifold or a torus is of this type. **It was conjectured that non-standard symplectic structures don't exist.**

THEOREM: (Entov-V.)
Let M be a compact even-dimensional torus, or a hyperkähler manifold (such as a K3 surface), and ω a standard symplectic form. **Then (M,ω) admits a full symplectic packing.**

REMARK: In this talk, **all tori are compact, even-dimensional, and satisfy $\dim_{\mathbb{R}} M \geqslant 4$.**

Motivation: Gromov Capacity

DEFINITION: Let M be a symplectic manifold. Define **Gromov capacity** $\mu(M)$ as the supremum of radii r, for all symplectic embeddings from a symplectic balls B_r to M.

DEFINITION: Define **symplectic volume** of a symplectic manifold (M, ω) as $\int_M \omega^{\frac{1}{2}\dim_{\mathbb{R}} M}$.

REMARK: Gromov capacity is obviously bounded by the symplectic volumes: a manifold of Gromov capacity r has volume $\geq \mathrm{Vol}(B_r)$. However, **there are manifolds of infinite volume with finite Gromov capacity.**

THEOREM: (Gromov)
Consider **a symplectic cylinder** $C_r := \mathbb{R}^{2n-2} \times B_r$ with the product symplectic structure. Then the Gromov capacity of C_r is r.

REMARK: This result was used by Gromov to study symplectic packing in $\mathbb{C}P^2$. He proved that **there is no full symplectic packing,** and found precise bounds.

Symplectic packing in $\mathbb{C}P^2$ (Gromov, McDuff, Polterovich, Biran)

THEOREM: Let v_N be a supremum of number V such that a collection of N equal symplectic balls of total volume V can be embedded to symplectic $\mathbb{C}P^2$ of volume 1. Then

N	1	2	3	4	5	6	7	8	9	$N > 9$
ν_N	1	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{20}{25}$	$\frac{24}{25}$	$\frac{63}{64}$	$\frac{288}{289}$	1	1

The first few numbers are due to Gromov, last to Biran, the rest are McDuff-Polterovich.

REMARK: **These numbers are related to Nagata conjecture**, which is still unsolved (Biran used Taubes' work on Seiberg-Witten invariants to avoid proving it).

CONJECTURE: Suppose $p_1, ..., p_r$ are very general points in $\mathbb{C}P^2$ and that $m_1, ..., m_r$ are given positive integers. **Then for any $r > 9$ any curve C in P2 that passes through each of the points p_i with multiplicity m_i must satisfy** $\deg C > \frac{1}{\sqrt{r}} \sum_{i=1}^r m_i$.

REMARK: Nagata conjecture was known already to Nagata when r is a full square, **and unknown for all other r, even when $p_1, ..., p_r$ are generic.**

Ekeland-Hofer theorem

THEOREM: (Ekeland-Hofer)
Let M, N be symplectic manifolds, and $\varphi: \ M \longrightarrow N$ a diffeomorphism. Suppose that for all sufficiently small, convex open sets $U \subset M$, Gromov capacity satisfies $\mu(U) = \mu(\varphi(U))$. **Then φ is a symplectomorphism.**

REMARK: This can be used to define C^0- **(continuous) symplectomorphisms.**

REMARK: Ekeland-Hofer theorem implies a theorem of Gromov-Eliashberg: **symplectomorphism group is C^0-closed in the group of diffeomorphisms.**

McDuff and Polterovich for Kähler manifolds

DEFINITION: Let M be a symplectic manifold, $x_1, ..., x_n \in M$ distinct points, and $r_1, ..., r_n$ a set of positive numbers. We say that M **admits symplectic packing** with centers $x_1, ..., x_n$ and radii $r_1, ..., r_n$ if there exists a symplectic embedding from a disconnected union of symplectic balls of radii $r_1, ..., r_n$ to M mapping centers of balls to $x_1, ..., x_n$.

THEOREM: (McDuff, Polterovich, 1995)
Let (M, ω) be a Kähler manifold, $\tilde{M} \stackrel{\nu}{\longrightarrow} M$ its blow-up in $x_1, ..., x_n$, E_i the corresponding exceptional divisors, and $[E_i]$ their fundamental classes. Assume that the class $\nu^* \omega - \sum_i c_i [E_i]$ is Kähler, for some $c_i > 0$. **Then M admits a symplectic packing with radii $r_i = \pi^{-1} \sqrt{c_i}$.**

McDuff and Polterovich for tamed manifold

DEFINITION: An almost complex structure I on M **is tamed by a symplectic form** $\omega \in \Lambda^2 M$ if $\omega(x, Ix) > 0$ for any non-zero tangent vector $x \in TM$.

THEOREM: (McDuff-Polterovich, 1995) Let (M,ω) be a compact symplectic manifold, $\tilde{M} \stackrel{\nu}{\longrightarrow} M$ its symplectic blow-up in $x_1, ..., x_n$, E_i the corresponding exceptional divisors, $[E_i]$ their fundamental classes, and $r_1, \ldots, r_k$ a collection of positive numbers. Assume there exists an almost complex structure I of on M tamed by ω and a symplectic form $\tilde{\omega}$ on $\tilde{M}$ taming the pullback almost complex structure $\tilde{I}$ so that $[\tilde{\omega}] = \nu^*[\omega] - \pi \sum_{i=1}^k r_i^2 [E_i]$. **Then** (M,ω) **admits a symplectic embedding of** $\coprod_{i=1}^k B^{2n}(r_i)$**.** ■

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

COROLLARY: The group $SU(2)$ of orthogonal quaternions acts on triples (I, J, K) producing new hyperkähler structures.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Calabi-Yau and Bogomolov decomposition theorem

REMARK: A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

CLAIM: A compact hyperkähler manifold M **has maximal holonomy of Levi-Civita connection** $Sp(n)$ if and only if $\pi_1(M) = 0$, $h^{2,0}(M) = 1$.

THEOREM: (Bogomolov decomposition)
Any compact hyperkähler manifold has a finite covering isometric to a product of a torus and several maximal holonomy hyperkähler manifolds.

Campana simple manifolds

DEFINITION: A complex manifold M, $\dim_{\mathbb{C}} M > 1$, is called **Campana simple** if the union $\mathfrak{U}$ of all complex subvarieties $Z \subset M$ satisfying $0 < \dim Z < \dim M$ has measure 0. A point which belongs to $M \backslash \mathfrak{U}$ is called **generic**.

REMARK: **Campana simple manifolds are non-algebraic**. Indeed, a manifold which admits a globally defined meromorphic function f is a union of zero divisors for the functions $f-a$, for all $a \in \mathbb{C}$, and the zero divizor for f^{-1}. Hence **Campana simple manifolds admit no globally defined meromorphic functions**.

EXAMPLE: **A general complex torus has no non-trivial complex subvarieties,** hence it is Campana simple.

EXAMPLE: Let (M,I,J,K) be a hyperkähler manifold, and $L = aI+bJ+cK$, $a^2+b^2+c^2 = 1$ be a complex structure induced by quaternions. Then for all such (a,b,c) outside of a countable set, **all complex subvarieties $Z \subset (M,L)$ are hyperkähler, and (unless M a finite quotient of a product) $\bigcup_Z Z \neq M$** (V., 1994, 1996). **Therefore, (M,L) is Campana simple.**

CONJECTURE: (Campana)
Let M be a Campana simple Kähler manifold. **Then M is bimeromorphic to a finite quotient of a hyperkähler orbifold or a torus.**

Demailly-Paun theorem

REMARK: Let M be a compact Kähler manifold. Recall that the cohomology space $H^2(M,\mathbb{C})$ is decomposed as $H^2(M,\mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$ with $H^{1,1}(M)$ identified with the space of I-invariant harmonic 2-forms, and $H^{2,0}(M) \oplus H^{0,2}(M)$ the space of I-antiinvariant harmonic 2-forms. This decomposition is called **Hodge decomposition**. The space $H^{1,1}(M)$ is a complexification of a real space $H^{1,1}(M,\mathbb{R}) = \{\nu \in H^2(M,\mathbb{R}) \ \mid \ I(\nu) = \nu\}$.

THEOREM: (Demailly-Păun, 2002)
Let M be a compact Kähler manifold, and $\widehat{K}(M) \subset H^{1,1}(M,\mathbb{R})$ a subset consisting of all (1,1)-forms η which satisfy $\int_Z \eta^k > 0$ for any k-dimensional complex subvariety $Z \subset M$. **Then the Kähler cone of M is one of the connected components of $\widehat{K}(M)$.** ■

Kähler cone for blow-ups of Campana simple manifolds

Theorem 2: Let M be a Campana simple compact Kähler manifold, and $x_1, ..., x_n$ distinct generic points of M. Consider the blow-up $\tilde M$ of M in $x_1, ..., x_n$, let E_i be the corresponding blow-up divisors, and $[E_i] \in H^2(M, \mathbb{Z})$ its fundamental classes. Decompose $H^{1,1}(\tilde M, \mathbb{R})$ as $H^{1,1}(\tilde M, \mathbb{R}) = H^{1,1}(M, \mathbb{R}) \oplus \bigoplus \mathbb{R}[E_i]$. Assume that η_0 is a Kähler class on M. **Then for any $\eta = \eta_0 + c_i[E_i]$, the following conditions are equivalent.**

(i) η is Kähler on $\tilde M$.

(ii) all c_i are negative, and $\int_M \eta^{\dim_{\mathbb{C}} M} > 0$.

Proof of (ii) $\Rightarrow$ (i). Step 1:
All proper complex subvarieties of $\tilde M$ are either contained in E_i, or do not intersect E_i. The condition "η_0 is Kähler on M" implies $\int_Z \eta^k > 0$ for all subvarieties not intersecting E_i. Since $[E_i]$ restricted to E_i is $-[\omega_{E_i}]$, where ω_{E_i} is the Fubini-Study form, $c_i < 0$ implies that $\int_Z \eta^k > 0$ for all subvarieties which lie in E_i. Finally, the integral of η over $\tilde M$ is positive by the assumtion $\int_M \eta^{\dim_{\mathbb{C}} M} > 0$. **Therefore, the condition (ii) implies that $\eta \in \hat K(\tilde M)$.**

Kähler cone for blow-ups of Campana simple manifolds (cont.)

Theorem 2: Let M be a Campana simple compact Kähler manifold, and $x_1, ..., x_n$ distinct generic points of M. Consider the blow-up $\tilde{M}$ of M in $x_1, ..., x_n$, let E_i be the corresponding blow-up divisors, and $[E_i] \in H^2(M, \mathbb{Z})$ its fundamental classes. Decompose $H^{1,1}(\tilde{M}, \mathbb{R})$ as $H^{1,1}(\tilde{M}, \mathbb{R}) = H^{1,1}(M, \mathbb{R}) \oplus \bigoplus \mathbb{R}[E_i]$. Assume that η_0 is a Kähler class on M. **Then for any $\eta = \eta_0 + c_i[E_i]$, the following conditions are equivalent.**

(i) η is Kähler on $\tilde{M}$.

(ii) all c_i are negative, and $\int_M \eta^{\dim_{\mathbb{C}} M} > 0$.

Proof of (ii) $\Rightarrow$ (i). Step 2:

The form η_0 is Kähler on M, hence it lies on the boundary of the Kähler cone of $\tilde{M}$, and η_0 can be obtained as a limit

$$\eta_0 = \lim_{\varepsilon \to 0} \eta_0 + \varepsilon c_i[E_i]$$

of forms which lie in the same connected component of $\hat{K}(\tilde{M})$. Therefore, η **belongs to the same connected component of $\hat{K}(\tilde{M})$ as a Kähler form.** By Demailly-Păun, this implies that η is Kähler.

Proof of (i) $\Rightarrow$ (ii).

The numerical conditions of (ii) mean that $\eta \in \hat{K}(\tilde{M})$, hence they are satisfied automatically, as follows from Step 1. ■

Campana simple manifolds and symplectic packings

DEFINITION: Let M be a compact symplectic manifold of volume V. We say that M **admits a full symplectic packing** if for any disconnected union S of symplectic balls of total volume less than V, S admits a symplectic embedding to M.

Theorem 3: Let (M, I, ω_0) be a Kähler, compact, Campana simple manifold. **Then M admits a full symplectic packing.**

Proof. Step 1: Let $x_1, ..., x_n$ distinct generic points of M. Consider the blow-up $\tilde{M}$ of M in $x_1, ..., x_n$, let E_i be the corresponding blow-up divisors, and $[E_i] \in H^2(M, \mathbb{Z})$ their fundamental classes. As follows from McDuff-Polterovich, existence of full symplectic packing on M is implied by existence of a Kähler form $\omega(c_1, ..., c_n)$ on $\tilde{M}$ with cohomology class $[\omega(c_1, ..., c_n)] = [\omega_0] - \sum c_i[E_i]$ for all $(c_1, ..., c_n)$ satisfying $\int_{\tilde{M}}([\omega_0] - \sum c_i[E_i])^n > 0$

Step 2: Such a form exists by Theorem 2. ■

Symplectic packing on hyperkähler manifolds and compact tori with irrational symplectic form

DEFINITION: A symplectic form is called **irrational** if its cohomology class is irrational, that is, lies in $H^2(M,\mathbb{R})\backslash\mathbb{R}\cdot H^2(M,\mathbb{Q})$.

THEOREM: Let M be a hyperkähler manifold or a compact torus, ω an irrational, standard symplectic form, and $\mathcal{T}$ the set of complex structures for which ω is Kähler. **Then the set $\mathcal{T}_0\subset\mathcal{T}$ of Campana simple complex structures is dense in $\mathcal{T}$ and has full measure in the corresponding moduli space.**

Proof: The Hodge loci of hyperkähler manifolds admitting a non-hyperkähler subvariety have positive codimension, and deformations of hyperkähler subarieties never cover M. ■

COROLLARY: Let M be a hyperkähler manifold or a compact torus, equipped with a standard, irrational symplectic form ω. **Then M admits full symplectic packing.**

Proof: By definition of a standard symplectic form, there exists a complex structure I such that ω is Kähler. Deforming I in $\mathcal{T}$, we obtain a Campana simple complex structure for which ω is Kähler. Then (M,ω) admits full symplectic packing by Theorem 3. ■

Symplectic cone and Kähler cone

DEFINITION: An almost complex structure I **tames** a symplectic structure ω if $\omega_I^{1,1}$ is a Hermitian form on (M,I).

PROPOSITION: Let (M,I,ω) be an almost complex tamed symplectic manifold, and $\eta \in \Lambda^{2,0+0,2}(M,\mathbb{R})$ a closed real $(2,0)+(0,2)$-form. **Then $\omega+\eta$ is also a symplectic form.** Moreover, **the complex structure I is tamed by $\omega+\eta$.**

Proof. Step 1: Since $I(\eta)=-\eta$ for each $(2,0)+(0,2)$-form η, one has $\omega(x,Ix)=\omega^{1,1}(x,Ix)>0$ for each non-zero x.

Step 2: Since $\eta^{1,1}=0$, one has $\omega+\eta(x,Ix)=\omega^{1,1}(x,Ix)>0$ for each non-zero x. Therefore, $\omega+\eta$ is non-degenerate. ■

DEFINITION: A **symplectic class** of a manifold M is a cohomology class of a symplectic form on M. **Symplectic cone** of a symplectic manifold M is a set $\operatorname{Symp}(M) \subset H^2(M,\mathbb{R})$ of all symplectic classes. **Taming cone** of (M,I) is a cone of symplectic classes of all symplectic form taming an almost complex structure I.

Corollary 1: Let M be a Kähler manifold, and $\operatorname{Kah}(M)$ its Kähler cone. **Then the taming cone of M contains $\operatorname{Kah}(M)+H^{2,0+0,2}(M,\mathbb{R})$.** ■

Symplectic cone for blown-up tori and hyperkähler manifolds

Theorem 4: Let (M, I, ω_I) be a compact Kähler manifold obtained as a limit of Campana simple manifolds. **Then (M, ω_I) admits full symplectic packing.**

Proof. Step 1: Let B be an open neighbourhood of I in the moduli space of complex structures on M, and $B \stackrel{\varphi}{\longrightarrow} H^2(M, \mathbb{R})$ a map putting J to $(\omega_I)^{1,1}_J$. By Kodaira stability theorem, $\varphi(J)$ is a Kähler class for J sufficiently close to I. Therefore, **there exists a Campana simple complex structure J such that $\omega_J := (\omega_I)^{1,1}_J$ is Kähler, arbitrarily close to I in B.**

Step 2: By Theorem 3, $\eta_J := \omega_J + \sum c_i[E_i]$ **is a Kähler class on a blow-up of (M, J), with blow-up points generic.** Indeed, the condition $\int_M \eta_J^{\dim_{\mathbb{C}} M} > 0$ remains true for J sufficiently close to I.

Step 3: Now, $\omega_I - \omega_J$ is by definition a $(2,0) + (0,2)$-cohomology class on (M, J). Therefore, $\eta = \omega_I + \sum c_i[E_i]$ is obtained from a Kähler form η_J by adding a $(2,0)+(0,2)$-form on (M, J). This implies that η **belongs to taming cone,** and Theorem 4 follows from the taming version of McDuff-Polterovich.

■

Further directions

1. We explored symplectic packing by symplectic balls. What about a packing by other subsets $K \subset \mathbb{R}^{2n}$?

1A. Define a packing number $\nu(K, M)$ of (K, ω) to M as a supremum of all ε for which $(K, \varepsilon\omega)$ admits a symplectic embedding to M. This function is obviously semicontinuous on K and M. When K is a union of symplectic balls, and M a hyperkähler manifold or a torus, $\nu(K, M) = \frac{\operatorname{Vol}(M)}{\operatorname{Vol}(K)}$. Using ergodicity, it is possible to show that $\frac{\nu(K,M)}{\operatorname{Vol}(M)}$ is constant for irrational symplectic structures on such M. Is it equal to 1? If so, we have "full packing by K".

2. Replacing blow-ups by orbifold blow-ups and balls by symplectic ellipsoids with rational axis length, our argument would give full packing by ellipsoids.

3. Let Symp be the infinite-dimensional Frechet manifold of all symplectic forms on M, and Diff the diffeomorphism group. The full packing phenomena seems to be related to ergodicity of Diff-action on Symp: the packing defines a semi-continuous, Diff-invariant function on Symp, which should be a posteriori constant on the set of all symplectic structures with dense Diff-orbits. One could study other semi-continuous quantities in relation to Diff-action and ergodicity.

APPENDIX: WON'T BE SHOWN

unless demanded

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I : TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X, Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

Kähler manifolds

DEFINITION: A Riemannian metric g on an almost complex manifold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2 y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure I is integrable, and the Hermitian form ω is closed.

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M. The set of all Kähler classes is called **the Kähler cone**.

Kähler structure on a blow-up

DEFINITION: Let S be a total space of a line bundle $\mathcal{O}(-1)$ on $\mathbb{C}P^n$, identified with a space of pairs $(z \in \mathbb{C}P^n, t \in z)$, where t is a point on a line $z \subset \mathbb{C}^{n+1}$ representing z. The forgetful map $\pi: S \longrightarrow \mathbb{C}^{n+1}$ is called **a blow-up of $\mathbb{C}^{n+1}$ in 0**. Given an open ball $B \subset \mathbb{C}^{n+1}$, the map $\pi: \pi^{-1}(B) \longrightarrow B$ is called **a blow-up of B in 0**. To blow up a point in a complex manifold M, we remove a ball B around this point, and replace it with a blown-up ball $\tilde{B}$, gluing $B\backslash x \subset \tilde{B}$ with $B\backslash x \subset M$.

PROBLEM: Suppose that M is Kähler, and $\tilde{M}$ is its blow-up. **Find a Kähler metric on $\tilde{M}$ and write it explicitly.**

Answer: **Symplectic blow-up!**

REMARK: In this talk, I would often drop all π and other constants from the equations.

Symplectic quotient

DEFINITION: Let ρ be an S^1-action on a symplectic manifold (M,ω) preserving the symplectic structure, and $\vec{v}$ its unit tangent vector. Cartan's formula gives $0 = \operatorname{Lie}_{\vec{v}}\omega = d(\omega \lrcorner \vec{v})$, hence $\omega \lrcorner \vec{v}$ is a closed 1-form. **Hamiltonian**, or **moment map** of ρ is an S^1-invariant function μ such that $d\mu = \omega \lrcorner \vec{v}$, and **symplectic quotient** $M/\!\!/_c S^1$ is $\mu^{-1}(c)/S^n$.

REMARK: In these assumptions, restriction of the symplectic form ω to $\mu^{-1}(c)$ vanishes on $\vec{v}$, hence it is **obtained as a pullback of a closed 2-form $\omega_{/\!\!/}$ on $M/\!\!/_c S^1$.**

THEOREM: The form $\omega_{/\!\!/}$ is a symplectic form on $M/\!\!/_c S^1$. In other words, **the symplectic quotient is a symplectic manifold.**

REMARK: If, in addition, M is equipped with a Kähler structure (I,ω), and S^1-action preserves the complex structure, the symplectic quotient $M/\!\!/_c S^1$ inherits the Kähler structure. In this case it is called **a Kähler quotient**. Whenever the S^1-action can be integrated to holomorphic $\mathbb{C}^*$-action, **the Kähler quotient is identified with an open subset of its orbit space.**

REMARK: The moment map is defined by $d\mu = \omega \lrcorner \vec{v}$ uniquely up to a constant. However, the symplectic quotient $M/\!\!/_c S^1 = \mu^{-1}(c)/S^n$ **depends heavily on the choice of $c \in \mathbb{R}$.**

Symplectic blow-up

CLAIM: Consider the standard S^1-action on $\mathbb{C}^n$, and let $W \subset \mathbb{C}^n$ be an S^1-invariant open subset. Consider the product $V := W \times \mathbb{C}$ with the standard symplectic structure and take the S^1-action on $\mathbb{C}$ opposite to the standard one. **Then its moment map is $w - t$, where $w(x) = |x|^2$ is the length function on W and $r(t) = |t|^2$ the length function on $\mathbb{C}$.**

DEFINITION: **Symplectic cut** of W is $(W \times \mathbb{C})/\!\!/_c S^1$.

REMARK: Geometrically, the symplectic cut is obtained as follows. Take $c \in \mathbb{R}$, and let $W_c := \{w \in W \ \ | \ \ |w|^2 \leqslant c\}$. Then W_c is a manifold with boundary ∂W_c, which is a sphere $|w|^2 = c$. Then $(W \times \mathbb{C})/\!\!/_c S^1 = (W_c \times \mathbb{C})/\!\!/_c S^1$ is obtained from W_c by gluing each S^1-orbit which lies on ∂W_c to a point. Combinatorially, $(W \times \mathbb{C})/\!\!/_c S^1$ is $\mathbb{C}^n$ with 0 replaced with $\mathbb{C}P^{n-1}$.

DEFINITION: In these assumptions, **symplectic blow-up** of radius $\lambda = \sqrt{c}$ of W in 0 is $(W \times \mathbb{C})/\!\!/_c S^1$. **Symplectic blow-up** of a symplectic manifold M is obtained by removing a symplectic ball W and gluing back a blown-up symplectic ball $(W \times \mathbb{C})/\!\!/_c S^1$.

REMARK: The **symplectic form ω_c on the blow-up $(W \times \mathbb{C})/\!\!/_c S^1$ depends on c as follows:** $\int_l \omega_c = c$, where $l \subset E$ is a rational line on an exceptional divisor $E := \pi^{-1}(c)$.

McDuff and Polterovich: symplectic packing from symplectic blow-ups

DEFINITION: Let M be a symplectic manifold, $x_1, ..., x_n \in M$ distinct points, and $r_1, ..., r_n$ a set of positive numbers. We say that M **admits symplectic packing** with centers $x_1, ..., x_n$ and radii $r_1, ..., r_n$ if there exists a symplectic embedding from a disconnected union of symplectic balls of radii $r_1, ..., r_n$ to M mapping centers of balls to $x_1, ..., x_n$.

REMARK: The choice of x_i is irrelevant, because **the group of symplectic authomorphisms acts on M infinitely transitively.**

Theorem 1: (McDuff-Polterovich)
Let (M, ω) be a symplectic manifold, $x_1, ..., x_n \in M$ distinct points, and $c_1, ..., c_n$ a set of positive numbers. Let $\pi : \tilde{M} \longrightarrow M$ be a symplectic blow-up with centers in x_i, and $E_i \in H^2(\tilde{M}, \mathbb{Z})$ the fundamental classes of its exceptional divisors. Then the following conditions are equivalent.

(i) M **admits a symplectic packing with radii** $r_i = \pi^{-1}\sqrt{c_i}$

(ii) For any $\varepsilon \in [0, 1]$, **there exists a form $\omega_\varepsilon(c_1, ..., c_n)$ cohomologically equivalent to** $\pi^*\omega - \sum \varepsilon\pi c_i E_i$, symplectic for $\varepsilon > 0$, smoothly depending on ε, and satisfying $\omega_0(c_1, ..., c_n) = \pi^*\omega$. ■

McDuff and Polterovich for Kähler manifolds

REMARK: In Kähler situation, the smooth dependence condition is trivial, because for any two Kähler forms ω, ω', straight interval connecting ω to ω' consists of Kähler forms (indeed, **the set of Kähler forms is convex**). This brings the following corollary.

Corollary 1: Let (M, ω) be a Kähler manifold, $\tilde{M} \stackrel{\pi}{\longrightarrow} M$ its blow-up in $x_1, ..., x_n$, E_i the corresponding exceptional divisors, and $[E_i]$ their fundamental classes. Assume that the class $\pi^*\omega - \sum_i c_i[E_i]$ is Kähler, for some $c_i > 0$. **Then M admits a symplectic packing with radii $r_i = \pi^{-1}\sqrt{c_i}$.**