Symplectic packing for simple Kähler manifolds, hyperkähler manifolds and tori

Misha Verbitsky

Skolkovo Institute of Science and Technology
International Conference
“Geometry, Topology and Integrability”

October 21, 2014.
Symplectic packing

**DEFINITION:** A *symplectic ball* is a ball of radius $r$ in $\mathbb{R}^{2n}$, equipped with a standard symplectic structure $\omega = \sum dp_i \wedge dq_i$.

**DEFINITION:** Let $M$ be a compact symplectic manifold of volume $V$. We say that $M$ *admits a full symplectic packing* if for any disconnected union $S$ of symplectic balls of total volume less than $V$, $S$ admits a symplectic embedding to $M$.

**Definition 1:** Let $M$ be a torus. A symplectic structure $\omega$ is called *standard* if there exists a flat torsion-free connection preserving $\omega$.

**DEFINITION:** Let $M$ be a hyperkähler manifold. A symplectic structure $\omega$ is called *standard* if $\omega$ is a Kähler form for some hyperkähler structure.

**REMARK:** Any known symplectic structure on a hyperkähler manifold or a torus is of this type. It was conjectured that non-standard symplectic structures don’t exist.
**THEOREM:** (Entov-V.)

Let $M$ be a compact even-dimensional torus, $\dim_{\mathbb{R}} M \geq 4$ or a hyperkähler manifold (such as a K3 surface), and $\omega$ a standard symplectic form. Then $(M, \omega)$ admits a full symplectic packing.

**REMARK:** In this talk, all tori are compact, even-dimensional, and satisfy $\dim_{\mathbb{R}} M \geq 4$. 
Motivation: Gromov Capacity

**DEFINITION:** Let $M$ be a symplectic manifold. Define **Gromov capacity** $\mu(M)$ as the supremum of radii $r$, for all symplectic embeddings from a symplectic balls $B_r$ to $M$.

**DEFINITION:** Define **symplectic volume** of a symplectic manifold $(M, \omega)$ as $\int_M \omega^{\frac{1}{2} \dim M}$.

**REMARK:** Gromov capacity is obviously bounded by the symplectic volumes: a manifold of Gromov capacity $r$ has volume $\geq \text{Vol}(B_r)$. However, there are manifolds of infinite volume with finite Gromov capacity.

**THEOREM:** (Gromov)
Consider a **symplectic cylinder** $C_r := \mathbb{R}^{2n-2} \times B_r$ with the product symplectic structure. Then the Gromov capacity of $C_r$ is $r$.

**REMARK:** This result was used by Gromov to study symplectic packing in $\mathbb{C}P^2$. He proved that **there is no full symplectic packing**, and found precise bounds.
Ekeland-Hofer theorem

**THEOREM:** (Ekeland-Hofer)
Let $M$, $N$ be symplectic manifolds, and $\varphi : M \to N$ a diffeomorphism. Suppose that for all sufficiently small, convex open sets $U \subset M$, Gromov capacity satisfies $\mu(U) = \mu(\varphi(U))$. Then $\varphi$ is a symplectomorphism.

**REMARK:** This can be used to define $C^0$- (continuous) symplectomorphisms.

**REMARK:** Ekeland-Hofer theorem implies a theorem of Gromov-Eliashberg: symplectomorphism group is $C^0$-closed in the group of diffeomorphisms.
Kähler manifolds

**DEFINITION:** A Riemannian metric $g$ on an almost complex manifold $M$ is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2 y) = -g(y,Ix)$, hence $\omega(x,y) := g(x, Iy)$ is skew-symmetric.

**DEFINITION:** The differential form $\omega \in \Lambda^{1,1}(M)$ is called the **Hermitian form** of $(M, I, g)$.

**THEOREM:** Let $(M, I, g)$ be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure $I$ is integrable, and the Hermitian form $\omega$ is closed.

(ii) One has $\nabla(I) = 0$, where $\nabla$ is the Levi-Civita connection

$$\nabla : \text{End}(TM) \rightarrow \text{End}(TM) \otimes \Lambda^1(M).$$

**DEFINITION:** A complex Hermitian manifold $M$ is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form $\omega$ is called the **Kähler class** of $M$. The set of all Kähler classes is called **the Kähler cone**.
**Kähler structure on a blow-up**

**DEFINITION:** Let $S$ be a total space of a line bundle $\mathcal{O}(-1)$ on $\mathbb{C}P^n$, identified with a space of pairs $(z \in \mathbb{C}P^n, t \in z)$, where $t$ is a point on a line $z \subset \mathbb{C}^{n+1}$ representing $z$. The forgetful map $\pi : S \longrightarrow \mathbb{C}^{n+1}$ is called a **blow-up of $\mathbb{C}^{n+1}$ in 0**. Given an open ball $B \subset \mathbb{C}^{n+1}$, the map $\pi : \pi^{-1}(B) \longrightarrow B$ is called a **blow-up of $B$ in 0**. To blow up a point in a complex manifold $M$, we remove a ball $B$ around this point, and replace it with a blown-up ball $\tilde{B}$, gluing $B \backslash x \subset \tilde{B}$ with $B \backslash x \subset M$.

**PROBLEM:** Suppose that $M$ is Kähler, and $\tilde{M}$ is its blow-up. **Find a Kähler metric on $\tilde{M}$ and write it explicitly.**

**Answer:** **Symplectic blow-up!**

**REMARK:** In this talk, I would drop all $\pi$ and other constants from the equations.
**Symplectic blow-up**

**CLAIM:** Consider the standard $S^1$-action on $\mathbb{C}^n$, and let $W \subset \mathbb{C}^n$ be an $S^1$-invariant open subset. Consider the product $V := W \times \mathbb{C}$ with the standard symplectic structure and take the $S^1$-action on $\mathbb{C}$ opposite to the standard one. Then its moment map is $w - t$, where $w(x) = |x|^2$ is the length function on $W$ and $r(t) = |t|^2$ the length function on $\mathbb{C}$.

**DEFINITION:** Symplectic cut of $W$ is $(W \times \mathbb{C})/\!/_{c}S^1$.

**REMARK:** Geometrically, the symplectic cut is obtained as follows. Take $c \in \mathbb{R}$, and let $W_c : \{ w \in W \mid |w|^2 \leq c \}$. Then $W_c$ is a manifold with boundary $\partial W_c$, which is a sphere $|w|^2 = c$. Then $(W \times \mathbb{C})/\!/_{c}S^1 = (W_c \times \mathbb{C})/\!/_{c}S^1$ is obtained from $W_c$ by gluing each $S^1$-orbit which lies on $\partial W_c$ to a point. Combinatorially, $(W \times \mathbb{C})/\!/_{c}S^1$ is $\mathbb{C}^n$ with 0 replaced with $\mathbb{C}P^{n-1}$.

**DEFINITION:** In these assumptions, symplectic blow-up of radius $\lambda = \sqrt{c}$ of $W$ in 0 is $(W \times \mathbb{C})/\!/_{c}S^1$. Symplectic blow-up of a symplectic manifold $M$ is obtained by removing a symplectic ball $W$ and gluing back a blown-up symplectic ball $(W \times \mathbb{C})/\!/_{c}S^1$.

**REMARK:** The symplectic form $\omega_c$ on the blow-up $(W \times \mathbb{C})/\!/_{c}S^1$ depends on $c$ as follows: $\int_l \omega_c = c$, where $l \subset E$ is a rational line on an exceptional divisor $E := \pi^{-1}(c)$.
McDuff and Polterovich: symplectic packing from symplectic blow-ups

**DEFINITION:** Let $M$ be a symplectic manifold, $x_1, \ldots, x_n \in M$ distinct points, and $r_1, \ldots, r_n$ a set of positive numbers. We say that $M$ admits **symplectic packing** with centers $x_1, \ldots, x_n$ and radii $r_1, \ldots, r_n$ if there exists a symplectic embedding from a disconnected union of symplectic balls of radii $r_1, \ldots, r_n$ to $M$ mapping centers of balls to $x_1, \ldots, x_n$.

**REMARK:** The choice of $x_i$ is irrelevant, because the group of symplectic authomorphisms acts on $M$ infinitely transitively.

**Theorem 1:** (McDuff-Polterovich)
Let $(M, \omega)$ be a symplectic manifold, $x_1, \ldots, x_n \in M$ distinct points, and $c_1, \ldots, c_n$ a set of positive numbers. Let $\pi : \tilde{M} \rightarrow M$ be a symplectic blow-up with centers in $x_i$, and $E_i \in H^2(\tilde{M}, \mathbb{Z})$ the fundamental classes of its exceptional divisors. Then the following conditions are equivalent.

(i) $M$ admits a symplectic packing with radii $r_i = \sqrt{c_i}$

(ii) For any $\varepsilon \in [0, 1]$, there exists a form $\omega_\varepsilon(c_1, \ldots, c_n)$ cohomologically equivalent to $\pi^*\omega - \sum \varepsilon \pi c_i E_i$, symplectic for $\varepsilon > 0$, smoothly depending on $\varepsilon$, and satisfying $\omega_0(c_1, \ldots, c_n) = \pi^*\omega$. ■
McDuff and Polterovich for Kähler manifolds

REMARK: In Kähler situation, the smooth dependence condition is trivial, because for any two Kähler forms \( \omega, \omega' \), straight interval connecting \( \omega \) to \( \omega' \) consists of Kähler forms (indeed, the set of Kähler forms is convex). This brings the following corollary.

Corollary 1: Let \((M, \omega)\) be a Kähler manifold, \( \tilde{M} \xrightarrow{\pi} M \) its blow-up in \( x_1, ..., x_n, E_i \) the corresponding exceptional divisors, and \([E_i]\) their fundamental classes. Assume that the class \( \pi^* \omega - \sum c_i [E_i] \) is Kähler, for some \( c_i > 0 \). Then \( M \) admits a symplectic packing with radii \( r_i = \sqrt{c_i} \).
Hyperkähler manifolds

**DEFINITION:** A hyperkähler structure on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**REMARK:** This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves $I, J, K$.

**COROLLARY:** The group $SU(2)$ of orthogonal quaternions acts on triples $(I, J, K)$ producing new hyperkähler structures.

**DEFINITION:** Let $M$ be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called the holonomy group of $M$.

**REMARK:** A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving $I, J, K$).
**Calabi-Yau and Bogomolov decomposition theorem**

**REMARK:** A hyperkähler manifold is holomorphically symplectic: \( \omega_J + \sqrt{-1} \omega_K \) is a holomorphic symplectic form on \((M, I)\).

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

**CLAIM:** A compact hyperkähler manifold \( M \) has maximal holonomy of Levi-Civita connection \( Sp(n) \) if and only if \( \pi_1(M) = 0, \ h^{2,0}(M) = 1. \)

**THEOREM:** (Bogomolov decomposition)
Any compact hyperkähler manifold has a finite covering isometric to a product of a torus and several maximal holonomy hyperkähler manifolds.
EXAMPLES.

**EXAMPLE:** Even-dimensional complex torus $\mathbb{C}^{2n}/\mathbb{Z}^{4n} = \mathbb{H}^{n}/\mathbb{Z}^{4n}$

**EXAMPLE:** Take a 2-dimensional complex torus $T$, then $T/\pm 1$ is an orbifold with 16 double points. Its resolution $\widetilde{T}/\pm 1$ is called a Kummer surface. It is holomorphically symplectic.

**DEFINITION:** A K3-surface is a deformation of a Kummer surface.

**THEOREM:** Any complex compact surface with $c_1(M) = 1$ and $H^1(M) = 0$ is isomorphic to K3. Moreover, it is hyperkähler.
Hilbert schemes

**DEFINITION:** A Hilbert scheme $M^{[n]}$ of a complex surface $M$ is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension $n$ over $\mathbb{C}$.

**REMARK:** A Hilbert scheme is obtained as a resolution of singularities of the symmetric power $\text{Sym}^n M$.

**THEOREM:** (Fujiki, Beauville) A Hilbert scheme of a holomorphically symplectic manifold is hyperkähler.

**EXAMPLE:** A Hilbert scheme of K3.

**EXAMPLE:** Let $T$ is a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, it is called a generalized Kummer variety.

**REMARK:** There are 2 more “sporadic” examples of compact hyperkähler manifolds, constructed by K. O’Grady. All known compact hyperkaehler manifolds are finite quotients of the products of these 2 and the three series: tori, Hilbert schemes of K3, and generalized Kummer.
Campana simple manifolds

**DEFINITION:** A complex manifold $M$, $\dim_{\mathbb{C}} M > 1$, is called **Campana simple** if the union $\mathcal{U}$ of all complex subvarieties $Z \subset M$ satisfying $0 < \dim Z < \dim M$ has measure 0. A point which belongs to $M \setminus \mathcal{U}$ is called **generic**.

**REMARK:** Campana simple manifolds are non-algebraic. Indeed, a manifold which admits a globally defined meromorphic function $f$ is a union of zero divisors for the functions $f - a$, for all $a \in \mathbb{C}$, and the zero divisor for $f^{-1}$. Hence Campana simple manifolds admit no globally defined meromorphic functions.

**EXAMPLE:** A general complex torus has no non-trivial complex subvarieties, hence it is Campana simple.

**EXAMPLE:** Let $(M, I, J, K)$ be a hyperkähler manifold, and $L = aI + bJ + cK$, $a^2 + b^2 + c^2 = 1$ be a complex structure induced by quaternions. Then for all such $(a, b, c)$ outside of a countable set, all complex subvarieties $Z \subset (M, L)$ are hyperkähler, and (unless $M$ a finite quotient of a product) $\bigcup_{Z} Z \neq M$ (V., 1994, 1996). Therefore, $(M, L)$ is Campana simple.

**CONJECTURE:** (Campana)
Let $M$ be a Campana simple Kähler manifold. Then $M$ is bimeromorphic to a finite quotient of a hyperkähler orbifold or a torus.
Demainly-Paun theorem

REMARK: Let $M$ be a compact Kähler manifold. Recall that the cohomology space $H^2(M, \mathbb{C})$ is decomposed as $H^2(M, \mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$ with $H^{1,1}(M)$ identified with the space of $I$-invariant harmonic 2-forms, and $H^{2,0}(M) \oplus H^{0,2}(M)$ the space of $I$-antiinvariant harmonic 2-forms. This decomposition is called Hodge decomposition. The space $H^{1,1}(M)$ is a complexification of a real space $H^{1,1}(M, \mathbb{R}) = \{ \nu \in H^2(M, \mathbb{R}) \mid I(\nu) = \nu \}$.

THEOREM: (Demainly-Păun, 2002)
Let $M$ be a compact Kähler manifold, and $\tilde{K}(M) \subset H^{1,1}(M, \mathbb{R})$ a subset consisting of all $(1,1)$-forms $\eta$ which satisfy $\int_Z \eta^k > 0$ for any $k$-dimensional complex subvariety $Z \subset M$. Then the Kähler cone of $M$ is one of the connected components of $\tilde{K}(M)$. ■
Kähler cone for blow-ups of Campana simple manifolds

**Theorem 2:** Let $M$ be a Campana simple compact Kähler manifold, and $x_1, \ldots, x_n$ distinct generic points of $M$. Consider the blow-up $\tilde{M}$ of $M$ in $x_1, \ldots, x_n$, let $E_i$ be the corresponding blow-up divisors, and $[E_i] \in H^2(M, \mathbb{Z})$ its fundamental classes. Decompose $H^{1,1}(\tilde{M}, \mathbb{R})$ as $H^{1,1}(\tilde{M}, \mathbb{R}) = H^{1,1}(M, \mathbb{R}) \oplus \bigoplus \mathbb{R}[E_i]$. Assume that $\eta_0$ is a Kähler class on $M$. Then for any $\eta = \eta_0 + c_i[E_i]$, the following conditions are equivalent.

(i) $\eta$ is Kähler on $\tilde{M}$.

(ii) all $c_i$ are negative, and $\int_M \eta^{\dim \mathbb{C}M} > 0$.

**Proof of (ii) $\Rightarrow$ (i). Step 1:**
All proper complex subvarieties of $\tilde{M}$ are either contained in $E_i$, or do not intersect $E_i$. The condition “$\eta_0$ is Kähler on $M$” implies $\int_Z \eta^k > 0$ for all subvarieties not intersecting $E_i$. Since $[E_i]$ restricted to $E_i$ is $-\omega_{E_i}$, where $\omega_{E_i}$ is the Fubini-Study form, $c_i < 0$ implies that $\int_Z \eta^k > 0$ for all subvarieties which lie in $E_i$. Finally, the integral of $\eta$ over $\tilde{M}$ is positive by the assumption $\int_M \eta^{\dim \mathbb{C}M} > 0$. Therefore, the condition (ii) implies that $\eta \in \tilde{K}(\tilde{M})$. 

17
Kähler cone for blow-ups of Campana simple manifolds (cont.)

**Theorem 2:** Let $M$ be a Campana simple compact Kähler manifold, and $x_1, \ldots, x_n$ distinct generic points of $M$. Consider the blow-up $\tilde{M}$ of $M$ in $x_1, \ldots, x_n$, let $E_i$ be the corresponding blow-up divisors, and $[E_i] \in H^2(M, \mathbb{Z})$ its fundamental classes. Decompose $H^{1,1}(\tilde{M}, \mathbb{R})$ as $H^{1,1}(\tilde{M}, \mathbb{R}) = H^{1,1}(M, \mathbb{R}) \oplus \bigoplus \mathbb{R}[E_i]$. Assume that $\eta_0$ is a Kähler class on $M$. Then for any $\eta = \eta_0 + c_i [E_i]$, the following conditions are equivalent.

(i) $\eta$ is Kähler on $\tilde{M}$.

(ii) all $c_i$ are negative, and $\int_M \eta^{\dim \mathbb{C} M} > 0$.

**Proof of (ii) $\Rightarrow$ (i). Step 2:**
The form $\eta_0$ is Kähler on $M$, hence it lies on the boundary of the Kähler cone of $\tilde{M}$, and $\eta_0$ can be obtained as a limit

$$\eta_0 = \lim_{\varepsilon \to 0} \eta_0 + \varepsilon c_i [E_i]$$

of forms which lie in the same connected component of $\hat{K}(\tilde{M})$. Therefore, $\eta$ belongs to the same connected component of $\hat{K}(\tilde{M})$ as a Kähler form. By Demailly-Păun, this implies that $\eta$ is Kähler.

**Proof of (i) $\Rightarrow$ (ii).**
The numerical conditions of (ii) mean that $\eta \in \hat{K}(\tilde{M})$, hence they are satisfied automatically, as follows from Step 1. ■

18
Campana simple manifolds and symplectic packings

**DEFINITION:** Let $M$ be a compact symplectic manifold of volume $V$. We say that $M$ admits a full symplectic packing if for any disconnected union $S$ of symplectic balls of total volume less than $V$, $S$ admits a symplectic embedding to $M$.

**Theorem 3:** Let $(M, I, \omega_0)$ be a Kähler, compact, Campana simple manifold. Then $M$ admits a full symplectic packing.

**Proof. Step 1:** Let $x_1, \ldots, x_n$ distinct generic points of $M$. Consider the blow-up $\tilde{M}$ of $M$ in $x_1, \ldots, x_n$, let $E_i$ be the corresponding blow-up divisors, and $[E_i] \in H^2(M, \mathbb{Z})$ their fundamental classes. As follows from McDuff-Polterovich, existence of full symplectic packing on $M$ is implied by existence of a Kähler form $\omega(c_1, \ldots, c_n)$ on $\tilde{M}$ with cohomology class $[\omega(c_1, \ldots, c_n)] = [\omega_0] - \sum c_i[E_i]$ for all $(c_1, \ldots, c_n)$ satisfying $\int_{\tilde{M}}([\omega_0] - \sum c_i[E_i])^n > 0$

**Step 2:** Such a form exists by Theorem 2. ■
Symplectic packing on hyperkähler manifolds and compact tori with irrational symplectic form

**DEFINITION:** A symplectic form is called *irrational* if its cohomology class is irrational, that is, lies in $H^2(M, \mathbb{R}) \setminus H^2(M, \mathbb{Q})$.

**THEOREM:** Let $M$ be a hyperkähler manifold or a compact torus, $\omega$ an irrational, standard symplectic form, and $\mathcal{T}$ the set of complex structures for which $\omega$ is Kähler. Then the set $\mathcal{T}_0 \subset \mathcal{T}$ of Campana simple complex structures is dense in $\mathcal{T}$ and has full measure in the corresponding moduli space.

**Proof:** The proof is based on Hodge theory and Chevalley theorem on tensor invariants of algebraic groups. ■

**COROLLARY:** Let $M$ be a hyperkähler manifold or a compact torus, equipped with a standard, irrational symplectic form $\omega$. Then $M$ admits full symplectic packing.

**Proof:** By definition of a standard symplectic form, there exists a complex structure $I$ such that $\omega$ is Kähler. Deforming $I$ in $\mathcal{T}$, we obtain a Campana simple complex structure for which $\omega$ is Kähler. Then $(M, \omega)$ admits full symplectic packing by Theorem 3. ■
Symplectic cone and Kähler cone

**DEFINITION:** An almost complex structure $I$ tames a symplectic structure $\omega$ if $\omega^{1,1}_I$ is a Hermitian form on $(M, I)$.

**PROPOSITION:** Let $(M, I, \omega)$ be an almost complex tamed symplectic manifold, and $\eta \in \Lambda^{2,0} + 0, 2(M, \mathbb{R})$ a closed real $(2, 0) + (0, 2)$-form. Then $\omega + \eta$ is also a symplectic form.

**Proof. Step 1:** Since $I(\eta) = -\eta$ for each $(2, 0) + (0, 2)$-form $\eta$, one has $\omega(x, Ix) = \omega^{1,1}_I(x, Ix) > 0$ for each non-zero $x$.

**Step 2:** Since $\eta^{1,1} = 0$, one has $\omega + \eta(x, Ix) = \omega^{1,1}_I(x, Ix) > 0$ for each non-zero $x$. Therefore, $\omega + \eta$ is non-degenerate. ■

**DEFINITION:** A **symplectic class** of a manifold $M$ is a cohomology class of a symplectic form on $M$. **Symplectic cone** of a symplectic manifold $M$ is a set $\text{Symp}(M) \subset H^2(M, \mathbb{R})$ of all symplectic classes.

**Corollary 1:** Let $M$ be a Kähler manifold, and $\text{Kah}(M)$ its Kähler cone. Then the symplectic cone of $M$ contains $\text{Kah}(M) + H^{2,0} + 0, 2(M, \mathbb{R})$. ■
Symplectic cone for blown-up tori and hyperkähler manifolds

**Theorem 4:** Let $\tilde{M} \to M$ be a blow-up of hyperkähler manifold or a torus in generic points, $\text{Kah}_S(\tilde{M}) := \text{Kah}(\tilde{M}) + H^{2,0} + H^{0,2}(\tilde{M}, \mathbb{R})$, and $KS(\tilde{M})$ the union of $\text{Kah}_S(\tilde{M})$ for all Kähler deformations of the complex structure on $M$. Then $KS(\tilde{M})$ contains the set of all $\eta \in H^{1,1}(\tilde{M}, \mathbb{R}) = H^{1,1}(M, \mathbb{R}) \oplus \bigoplus \mathbb{R}[E_i]$, satisfying $\eta = \omega_I + c_i[E_i]$ such that $\omega_I$ is a Kähler class on $M$, all $c_i$ are negative, and $\int_M \eta^\dim_{\mathbb{C}}M > 0$.

**Proof.** **Step 1:** Let $B$ be an open neighbourhood of $I$ in the moduli space of complex structures on $M$, and $B \xrightarrow{\varphi} H^2(M, \mathbb{R})$ a map putting $J$ to $(\omega_I)^{1,1}$. By Kodaira stability theorem, $\varphi(J)$ is a Kähler class for $J$ sufficiently close to $I$. Therefore, there exists a Campana simple complex structure $J$ such that $\omega_J := (\omega_I)^{1,1}_J$ is Kähler, arbitrarily close to $I$ in $B$.

**Step 2:** By Theorem 1, $\eta_J := \omega_J + c_i[E_i]$ is a Kähler class on a blow-up of $(M, J)$, with blow-up points generic. Indeed, the condition $\int_M \eta_J^\dim_{\mathbb{C}}M > 0$ remains true for $J$ sufficiently close to $I$.

**Step 3:** Now, $\omega_I - \omega_J$ is by definition a $(2,0) + (0,2)$-cohomology class on $(M, J)$. Therefore, $\eta$ is obtained from a Kähler form $\eta_J$ by adding a $(2,0) + (0,2)$-form.
Further directions

1. We explored symplectic packing by symplectic balls. What about a packing by other subsets of $\mathbb{R}^{2n}$?

1A. Define a packing number $\nu(K, M)$ of $(K, \omega)$ to $M$ as a supremum of all $\varepsilon$ for which $(K, \varepsilon\omega)$ admits a symplectic embedding to $M$. This function is obviously semicontinuous on $K$ and $M$. When $K$ is a union of symplectic balls, and $M$ a hyperkähler manifold or a torus, $\nu(K, M) = \frac{\text{Vol}(M)}{\text{Vol}(K)}$. Using ergodicity, it is possible to show that $\frac{\nu(K, M)}{\text{Vol}(M)}$ is constant for irrational symplectic structures on such $M$. Is it equal to 1? If so, we have “full packing by $K$”.

2. Replacing blow-ups by orbifold blow-ups and balls by symplectic ellipsoids with rational axis length, same argument would give full packing by ellipsoids.
Further directions (cont.)

3. Let $\text{Symp}$ be the infinite-dimensional Frechet manifold of all symplectic forms on $M$, and let $\text{Diff}$ be the diffeomorphism group. The full packing phenomena seems to be related to ergodicity of $\text{Diff}$-action on $\text{Symp}$: the packing defines a semi-continuous, $\text{Diff}$-invariant function on $\text{Symp}$, which should be a posteriori constant on the set of all symplectic structures with dense $\text{Diff}$-orbits. One could study other semi-continuous quantities in relation to $\text{Diff}$-action and ergodicity.

4. Let $\text{Diff}_0$ be the group of isotopies (connected component of $\text{Diff}$). The symplectic Teichmüller space $\text{Teich}_s := \text{Symp} / \text{Diff}_0$ is locally diffeomorphic to $H^2(M, \mathbb{R})$ by Moser’s theorem. The arguments similar to given above would show that standard connected components of $\text{Teich}_s$ are embedded to $H^2(M, \mathbb{R})$ (for hyperkähler manifolds), and the image is a defined by a single quadratic inequality. For a torus $M = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, the standard component of $\text{Teich}_s$ is a covering of the manifold $\{\omega \in \Lambda^2 \mathbb{R}^{2n} \mid \omega^n \neq 0\}$. Is this covering trivial?