Holomorphic bundles on hyperkahler manifolds and Tannakian categories

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Higgs bundles and related topics

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Tannakian categories

REMARK: All fields are assumed of $\operatorname{char} = 0$, an all abelian categories over a base field k.

DEFINITION: Tensor category is an abelian category equipped with the "tensor product" bifunctor $A, B \longrightarrow A \otimes B$, which is associative up to a natural isomorphism and admits a (left and right) unit object 1. It is called **symmetric** if the tensor product is commutative and **rigid** if there exists a contravariant functor $X \longrightarrow X^*$ and morphisms $1 \xrightarrow{\eta} X \otimes X^*$, $X^* \otimes X \xrightarrow{\varepsilon} 1$ defining isomorphisms

$$X = 1 \otimes X \xrightarrow{\eta} (X \otimes X^*) \otimes X = X \otimes (X^* \otimes X) \xrightarrow{\varepsilon} X$$

and

$$X^* = X^* \otimes 1 \xrightarrow{\eta} X^* \otimes (X \otimes X^*) = (X^* \otimes X) \otimes X^* \xrightarrow{\varepsilon} X^*.$$

DEFINITION: Symmetric rigid tensor category is called **Tannakian** if it is equipped with **fiber functor** to vector spaces which is exact, faithful and compatible with tensor product and duality.

"Tannaka-Krein duality"

THEOREM: (Saavedra, Deligne: "Tannaka-Krein duality") **Any Tannakian category is equivalent to the category of finite-dimensional representations of a pro-algebraic group** (**pro-algebraic group** is a projective limit of algebraic groups).

REMARK: The proof can be deduced directly from the following theorem.

THEOREM: (Chevalley) An algebraic group is determined by its projective invariants, in the followign sense. Let V be an exact representation of an algebraic group, $T^{\otimes i}(V \oplus V^*)$ the tensor power, and $Z \subset \prod_i \mathbb{P}T^{\otimes i}(V \oplus V^*)$ the set of G-invariant vectors in the projectivization of all tensor spaces. Then **G is the maximal subgroup of $GL(V)$ fixing all points of Z.**

Deligne, P., and Milne, J.S., Tannakian Categories, in Hodge Cycles, Motives, and Shimura Varieties, LNM 900, 1982, pp. 101-228", `http://www.jmilne.org/math/xnotes/tc.html`.

Pro-algebraic completion

DEFINITION: Let G be a group, $\mathcal{C}$ the category of all matrix representations $\rho: \ G \longrightarrow GL(V)$ of G, and G_V the Zariski closure of G in $GL(V)$. **Pro-algebraic completion** G_{pa} is the initial object of $\mathcal{C}$, that is, the limit of Zariski closures of images of G for all representations.

REMARK: Let G be a group and $\mathcal{C}$ category of all its representations. **Using the Tannaka-Krein duality, we can recover the proalgebraic completion of G.**

EXAMPLE: Category $\mathcal{C}_l$ of local systems on a manifold M is Tannakian.

EXAMPLE: Category $\mathcal{C}_u$ of unitary local systems on M is also Tannakian.

REMARK: The tautological functor $\mathcal{C}_u \longrightarrow \mathcal{C}_l$ defines the homomorphism of corresponding pro-algebraic completions $\pi_1(M)^l \longrightarrow \pi_1(M)^u$. Heuristically, this map is "Levi quotient", or "pro-reductive quotient": a quotient over the Levi subgroup. Indeed, all objects of $\mathcal{C}_u$ are semisimple, hence $\pi_1(M)^u$ is reductive.

Tensor category of instantons

DEFINITION: Let M be a 4-dimensional compact Riemannian manifold, $\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M)$ the standard decomposition, and (B, ∇) a bundle with connection such that its curvature belongs to $\Lambda^-(M) \otimes \operatorname{End}(B)$. Then (B, ∇) is called **ASD (anti-selfdual) bundle (Yang-Mills bundle, instanton).** Define category $\mathcal{C}_{ASD}$ of ASD bundles, with morphisms compatible with connection.

EXAMPLE: The category $\mathcal{C}_{ASD}$ is clearly Tannakian (any fiber can serve as the fiber functor).

QUESTION: Does it depend on the Riemannian metric g on M (say, when g is generic)?

REMARK: Category of unitary ASD bundles is independent from g if g is hyperkähler and "generic enough" (will be proven later).

Instantons and Higgs fields

The category $\mathcal{C}^u_{ASD}$ of (unitary) ASD bundles can be understood as "hyperkähler analogue of the category of unitary representation of the fundamental group".

The main subject of today's talk:

Construct Tannakian category $\mathcal{C}^{nu}_{ASD}$ related to $\mathcal{C}^u_{ASD}$ in the same way as category of all representations of fundamental group are related to unitary representation.

REMARK: Objects of $\mathcal{C}^{nu}_{ASD}$ should play the role of "Higgs bundles" in instanton category. In particular, their deformation spaces should be identified with complexification of the deformation spaces of objects of $\mathcal{C}^u_{ASD}$.

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the Levi-Civita connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

THEOREM: (Calabi-Yau)
A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

The $\overline{\partial}$-operator on vector bundles

DEFINITION: A $\overline{\partial}$-operator on a smooth bundle is a map $V \xrightarrow{\overline{\partial}} \Lambda^{0,1}(M) \otimes V$, satisfying $\overline{\partial}(fb) = \overline{\partial}(f) \otimes b + f\overline{\partial}(b)$ for all $f \in C^\infty M, b \in V$.

REMARK: A $\overline{\partial}$-operator on B can be extended to

$$\overline{\partial}: \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V,$$

using $\overline{\partial}(\eta \otimes b) = \overline{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \overline{\partial}(b)$, where $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

REMARK: If $\overline{\partial}$ is a holomorphic structure operator, then $\overline{\partial}^2 = 0$.

THEOREM: Let $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a $\overline{\partial}$-operator, satisfying $\overline{\partial}^2 = 0$. **Then $B := \ker \overline{\partial} \subset V$ is a holomorphic vector bundle of the same rank.**

DEFINITION: $\overline{\partial}$-operator $\overline{\partial}: V \longrightarrow \Lambda^{0,1}(M) \otimes V$ on a smooth manifold is called **a holomorphic structure operator**, if $\overline{\partial}^2 = 0$.

Connections and holomorphic structure operators

DEFINITION: let (B,∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}\ B \longrightarrow \Lambda^{0,1}(M)\otimes B$. Consider a Hodge decomposition $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1}:\ B \longrightarrow \Lambda^{0,1}(M)\otimes B, \quad \nabla^{1,0}:\ B \longrightarrow \Lambda^{1,0}(M)\otimes B.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: A Chern connection on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

REMARK: The curvature of a Chern connection on B is an End(B)-valued (1,1)-form: $\Theta_B \in \Lambda^{1,1}(\mathrm{End}(B))$.

REMARK: A converse is true, too. Given a Hermitian connection ∇ on a vector bundle B with curvature in $\Lambda^{1,1}(\mathrm{End}(B))$, we obtain a holomorphic structure operator $\overline{\partial} = \nabla^{0,1}$. Then, **$\nabla$ is a Chern connection of $(B,\overline{\partial})$.**

Hyperholomorphic connections

REMARK: Let M be a hyperkähler manifold. **The group $SU(2)$ of unitary quaternions acts on $\Lambda^*(M)$ multiplicatively.**

DEFINITION: A **hyperholomorphic connection** on a vector bundle B over M is a Hermitian connection with $SU(2)$-invariant curvature $\Theta \in \Lambda^2(M) \otimes \operatorname{End}(B)$.

REMARK: Since the invariant 2-forms satisfy $\Lambda^2(M)_{SU(2)} = \bigcap_{I\in \mathbb{C}P^1} \Lambda^{1,1}_I(M)$, **a hyperholomorphic connection defines a holomorphic structure on B** for each I induced by quaternions.

REMARK: Let M be a compact hyperkähler manifold. Then $SU(2)$ preserves harmonic forms, hence **acts on cohomology.**

CLAIM: All Chern classes of hyperholomorphic bundles are $SU(2)$-invariant.

Proof: Use $\Lambda^{2p}(M)_{SU(2)} = \bigcap_{I\in \mathbb{C}P^1} \Lambda^{p,p}_I(M)$. ■

REMARK: **Converse is also true** (for stable bundles). See the next slide.

Kobayashi-Hitchin correspondence

DEFINITION: Let F be a coherent sheaf over an n-dimensional compact Kähler manifold M. Let

$$\operatorname{slope}(F) := \frac{1}{\operatorname{rank}(F)} \int_M \frac{c_1(F) \wedge \omega^{n-1}}{vol(M)}.$$

A torsion-free sheaf F is called **(Mumford-Takemoto) stable** if for all subsheaves $F' \subset F$ one has $\operatorname{slope}(F') < \operatorname{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

DEFINITION: A Hermitian metric on a holomorphic vector bundle B is called **Yang-Mills** (Hermitian-Einstein) if the curvature of its Chern connection satisfies $\Theta_B \wedge \omega^{n-1} = \operatorname{slope}(F) \cdot \operatorname{Id}_B \cdot \omega^n$. A Yang-Mills connection is a Chern connection induced by the Yang-Mills metric.

REMARK: Yang-Mills connections minimize the integral

$$\int_M |\Theta_B|^2 \operatorname{Vol}_M$$

Kobayashi-Hitchin correspondence: (Donaldson, Uhlenbeck-Yau). Let B be a holomorphic vector bundle. **Then B admits Yang-Mills connection if and only if B is polystable.** Moreover such a connection is **unique**.

Kobayashi-Hitchin correspondence and hyperholomorphic bundles

CLAIM: Let M be a hyperkähler manifold. Then for any $SU(2)$-invariant 2-form $\eta \in \Lambda^2(M)$, one has $\eta \wedge \omega^{n-1} = 0$.

COROLLARY: Any bundle admitting hyperholomorphic connection is Yang-Mills, of slope 0 (and hence polystable).

REMARK: This implies that **a hyperholomorphic connection on a given holomorphic vector bundle is unique** (if exists). Such a bundle is called **hyperholomorphic**.

THEOREM: Let B be a polystable holomorphic bundle on (M, I), where (M, I, J, K) is hyperkähler. Then the (unique) **Yang-Mills connection on B is hyperholomorphic if and only if the cohomology classes $c_1(B)$ and $c_2(B)$ are $SU(2)$-invariant.**

COROLLARY: The moduli space of stable holomorphic vector bundles with $SU(2)$-invariant $c_1(B)$ and $c_2(B)$ **is a hyperkähler variety.**

COROLLARY: Let (M, I, J, K) be a hyperkähler manifold, and $L = aI + bJ + cK$ a generic induced complex structure. **Then any stable bundle on (M, L) is hyperholomorphic.**

Stable bundles on Kähler manifolds with trivial Picard group

We would now consider how the space of stable bundles changes when we change the hyperkähler structure.

DEFINITION: Let's call a coherent sheaf F **simple** if F has no subsheaves $F' \subset F$ with $0 < \operatorname{rk} F' < \operatorname{rk} F$.

CLAIM: Let (M, I) be a compact Kähler manifold with the trivial group $H^{1,1}(M, \mathbb{Z})$ of Hodge cycles (such a manifold is necessarily non-algebraic, because **it has no divisors**). Then **a coherent sheaf on M is stable if and only if it is simple, polystable if it is semisimple (direct sum of simple) and semisimple otherwise.**

Proof: Indeed, the slope of any coherent sheaf on (M, I) is 0. ■

Hyperkähler rotations

DEFINITION: Let (M, I, J, K, g) be a hyperkähler manifold, and $h \in \mathbb{H}$ a unitary quaternion. Then (M, hI, hJ, hK, g) is also a hyperkähler structure. This operation is called **hyperkähler rotation**.

REMARK: The orbit of I under all hyperkähler rotations is the set S of all complex structure of form $aI + bJ + cK$, $a^2 + b^2 + c^2 = 1$, identified with $S^2 = \mathbb{C}P^1$. It is called **twistor line** or **hyperkähler line**. It is not hard to see that S **represents a holomorphic curve in the Teichmüller space of complex structures on M.**

REMARK: Hyperkähler rotation **maps the category of hyperholomorphic bundles on (M, I) to hyperholomorphic bundles on (M, hI).**

HH-generic complex structures

DEFINITION: A complex structure I on a compact hyperkähler manifold is called **HH-generic** if $H_I^{1,1}(M,\mathbb{Z}) = 0$ and all vectors in $H_I^{2,2}(M,\mathbb{Z})$ are $SU(2)$-invariant with respect to all $SU(2)$-actions induced by unitary quaternions for all hyperkähler structures on M.

CLAIM: On a HH-generic manifold, a bundle is **polystable if and only if it is hyperholomorphic.** ■

THEOREM: Let I_1, I_2 be two complex structures on a hyperkähler manifold with $H_{\mathbb{Z}}^{1,1}(M, I_i) = 0$ related by the hyperkähler rotation. Denote by $\mathcal{C}_i$ the category of polystable holomorphic bundles on (M, I_i). **Then $\mathcal{C}_1$ is equivalent to $\mathcal{C}_2$.**

Proof: Polystable bundles on (M, I_i) are the same as hyperholomorphic, but the category of hyperholomorphic bundles remains the same under a hyperkähler rotation. ■

CLAIM: Every two HH-generic complex structures in the same deformation class can be connected by a sequence of hyperkähler rotations..

COROLLARY: The category of hyperholomorphic bundles on an HH-generic hyperkähler manifold **is a deformation invariant.**

Non-Hermitian hyperholomorphic connections

DEFINITION: Let (B, ∇) be a bundle with connection over a hyperkähler manifold. It is called **Non-Hermitian hyperholomorphic** (or **NHYM - non-Hermitian Yang-Mills**) if its curvature is $SU(2)$-invariant: $\nabla^2 \in \Lambda^2_{SU(2)}(M) \otimes \operatorname{End}(B)$

Properties of NHYM bundles which make it similar to flat bundles:

1. **Category fo NHYM bundles is a Tannakian category.**

2. The space S of NHYM connections is mapped to the space of holomorphic structures by the forgetful map (mapping the connection to its $\overline{\partial}$-operator. **There is a complex and holomorphic symplectic structure on S such that the forgetful map φ is a Lagrangian fibration.**

Twistor spaces

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors (Fujiki).

DEFINITION: A **twistor space** $\operatorname{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\operatorname{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ satisfies $I_{\operatorname{Tw}}^2 = -\operatorname{Id}$. **It defines an almost complex structure on $\operatorname{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

REMARK: For M compact, $\operatorname{Tw}(M)$ never admits a Kähler structure.

Twistor transform and hyperholomorphic bundles 1: direct twistor transform

CLAIM: Let $\sigma : \text{Tw}(M) \longrightarrow M$ be the standard projection, where M is hyperkähler or quaternionic-Kähler, and $\eta \in \Lambda^2 M$ a 2-form. Then **$\sigma^*\eta$ is a (1,1)-form iff η is $SU(2)$-invariant.**

COROLLARY: Let (B, ∇) be a bundle with connection, and $\sigma^* B, \sigma^* \nabla$ its pullback to Tw(M). **Then $(\sigma^* B, \sigma^* \nabla)$ has (1,1)-curvature iff ∇ has $SU(2)$-invariant curvature.**

REMARK: This construction produces a holomorphic vector bundle on Tw(M) starting from a NHYM bundle (bundle with connection with $SU(2)$-invariant curvature). It is called **direct twistor transform**. The **inverse twistor transform** produces a bundle with connection on M from a holomorphic bundle on Tw(M).

Twistor transform and hyperholomorphic bundles 2: inverse twistor transform

DEFINITION: Let M be a hyperkähler or quaternionic-Kähler manifold, and $\sigma: \operatorname{Tw}(M) \longrightarrow M$ its twistor space. For each point $x \in M$, $\sigma^{-1}(x)$ is a holomorphic rational curve in $\operatorname{Tw}(M)$. It is called **a horizontal twistor line**.

THEOREM: (The inverse twistor transform; Kaledin-V.) Let B be a holomorphic vector bundle on $\operatorname{Tw}(M)$, which is trivial on any horizontal twistor line. Denote by B_0 the C^∞-bundle on M with fiber $H^0(B\big|_{\sigma^{-1}(x)})$ at $x \in M$. **Then B_0 admits a unique non-Hermitian hyperholomorphic connection ∇** such that B is isomorphic (as a holomorphic vector bundle) to its twistor transform $(\sigma^* B_0, (\sigma^*\nabla)^{0,1})$.

REMARK: The condition of being trivial on any horizontal twistor line is **open.** Therefore, **a holomorphic bundle on a $\operatorname{Tw}(M)$ is "more or less the same" as a bundle with non-Hermitian hyperholomorphic connection on M.** In particular, geometric structures on the moduli of NHYM-connections (complex structure, etc) **are induced from the moduli of holomorphic bundles on $\operatorname{Tw}(M)$.**

Space of NHYM connections

THEOREM: The space of stable hyperholomorphic bundles on M is hyperkähler. Indeed, it has a complex structure for each complex structure on M induced by quaternions; **these complex structures satisfy quaternionic relations.** It might have singularities, but **they are resolved by normalization.**

THEOREM: Let M be a compact hyperkähler manifold, and NHYM(M) the space of all holomorphic bundles on Tw(M) which admit twistor transform (that is, are trivial on horizontal rational curves) and stable on each fiber of the projection π : Tw$(M) \longrightarrow \mathbb{C}P^1$. Denote by W the space of all stable hyperholomorphic bundles on M, considered as a hyperkähler variety, and let Sec(W) be the space of all holomorphic sections of the projection Tw$(W) \longrightarrow \mathbb{C}P^1$. **Then** NHYM$(M) =$ Sec(W)**.**

Proof: This is pretty much a tautology. ■

Semistable bundles and NHYM connections

Twistor correspondence can be used to prove the following result.

THEOREM: Let (M, I) be a HH-generic complex structure, and Bun(M, I) the category of vector bundles on (M, I), with all Jordan-Holder subquotients locally free. **Then Bun(M, I) is the same for all HH-generic I in the same deformation class.**

Proof: To see this, we consider the forgetful functor from a subcategory of NHYM(M) with all Jordan-Holder subquotients locally free to Bun(M, I) and show that it's surjective on objects and morphisms. This is used to identify Bun(M, I) with Bun(M, hI) for a hyperkähler rotation h. ■

Complexification of a hyperkähler manifold.

REMARK: Consider an anticomplex involution $\text{Tw}(M) \stackrel{\iota}{\longrightarrow} \text{Tw}(M)$ mapping (m,t) to $(m,i(t))$, where $i: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ is a central symmetry. Then $\text{Sec}_{hor}(M) = M$ **is a component of the fixed set of** ι**.**

COROLLARY: $\text{Sec}(M)$ **is a complexification of** M**.**

QUESTION: What are geometric structures on $\text{Sec}(M)$?

Answer 1: For compact M, a suitable partial compactification pf $\text{Sec}(M)$ **is holomorphically convex**
(Stein if $\dim M = 2$ or if M is a torus).

Answer 2: (Jardim-V.) The space $\text{Sec}_0(M)$ **admits a holomorphic, torsion-free connection** with holonomy $Sp(n,\mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.

Answer 3: (Jardim-V.) The space $\text{Sec}_0(M)$ **is equipped with a trisymplectic structure** (a triple of holomorphic symplectic structures which satisfy the same linear-algebraic relations as the triple of symplectic structures on a hyperkähler manifold).