# Teichmuüller space of symplectic packings

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## Symplectic packings

**DEFINITION:** A **symplectic ball** is a ball of radius $r$ in $\mathbb{R}^{2n}$, equipped with a standard symplectic structure $\omega = \sum dp_i \wedge dq_i$.

**DEFINITION:** **Symplectic embedding** from a ball to $M$ is an injective map which is compatible with the symplectic structure and smoothly extends to the boundary.

**DEFINITION:** **Symplectic packing**, or **symplectic packing by balls of radii** $r_1, ..., r_n$ is a symplectic embedding from a disconnected sum of symplectic balls of radii $r_1, ..., r_n$ to a symplectic manifold $M$.

**QUESTION:** Group of symplectomorphisms Symp$(M)$ acts on the set of all symplectic packings. **What are its orbits**?

**QUESTION:** Fix radii $r_1, ..., r_n$, and consider the set $\mathfrak{P}_{r_1,...,r_n}$ of all symplectic packings with these radii. **Is the action of** Symp **transitive on** $\mathfrak{P}_{r_1,...,r_n}$**?**

## Teichmüller space for symplectic structures

**DEFINITION:** Let $\Gamma(\Lambda^2 M)$ be the space of all 2-forms on a manifold $M$, and $\operatorname{Symp} \subset \Gamma(\Lambda^2 M)$ the space of all symplectic 2-forms. We equip $\Gamma(\Lambda^2 M)$ with $C^\infty$-topology of uniform convergence on compacts with all derivatives. Then $\Gamma(\Lambda^2 M)$ is a Frechet vector space, and Symp a Frechet manifold.

**DEFINITION:** Consider the group of diffeomorphisms, denoted Diff or $\operatorname{Diff}(M)$ as a Frechet Lie group, and denote its connected component ("group of isotopies") by $\operatorname{Diff}_0$. The quotient group $\Gamma := \operatorname{Diff} / \operatorname{Diff}_0$ is called **the mapping class group** of $M$.

**DEFINITION:** **Teichmüller space of symplectic structures on $M$** is defined as a quotient $\operatorname{Teich}_s := \operatorname{Symp} / \operatorname{Diff}_0$. The quotient $\operatorname{Teich}_s / \Gamma = \operatorname{Symp} / \operatorname{Diff}$, is called **the moduli space of symplectic structures.**

**REMARK:** In many cases $\Gamma$ acts on $\operatorname{Teich}_s$ with dense orbits, hence **the moduli space is not always well defined.**

**DEFINITION:** Two symplectic structures are called **isotopic** if they lie in the same orbit of $\operatorname{Diff}_0$, and **diffeomorphic** is they lie in the same orbit of Diff.

## Moser's theorem

**DEFINITION:** Define **the period map** $\mathsf{Per}: \ \mathsf{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ mapping a symplectic structure to its cohomology class.

**THEOREM: (Moser, 1965)**
The **Teichmüler space** $\mathsf{Teich}_s$ **is a manifold** (possibly, non-Hausdorff), and the **period map** $\mathsf{Per}: \ \mathsf{Teich}_s \longrightarrow H^2(M, \mathbb{R})$ **is locally a diffeomorphism.**

The proof is based on another theorem of Moser.

**Theorem 1: (Moser)**
Let $\omega_t$, $t \in S$ be a smooth family of symplectic structures, parametrized by a connected manifold $S$. Assume that the cohomology class $[\omega_t] \in H^2(M)$ is constant in $t$. **Then all $\omega_t$ are diffeomorphic.**

## The proof of Moser's theorem

**THEOREM: (Moser)**
The **Teichmüler space** $\mathrm{Teich}_s$ **is a manifold** (possibly, non-Hausdorff), and the **period map** $\mathrm{Per}: \ \mathrm{Teich}_s \longrightarrow H^2(M,\mathbb{R})$ **is locally a diffeomorphism.**

**Proof. Step 1:** We can locally find a section $S$ for the $\mathrm{Diff}_0$-action on Symp, producing a local decomposition $\mathrm{Symp} = O \times S$, where $O$ is a $\mathrm{Diff}_0$-orbit. Here $O$ and $S$ are both Frechet manifolds.

**Step 2:** The period map $P: \ U \longrightarrow H^2(M,\mathbb{R})$ is a smooth submersion. By Theorem 1, the fibers of $P$ are 0-dimensional. Therefore, $P$ is locally a diffeomorphism. ■

## Non-Hausdorff points on symplectic Teichmüller space

Example of D. McDuff found in Salamon, Dietmar, *Uniqueness of symplectic structures*, Acta Math. Vietnam. 38 (2013), no. 1, 123-144.

Let $M = S^1 \times Z^1 \times S^2 \times S^2$ with coordinates $\theta_1, \theta_2 \in S^1 \subset \mathbb{C}^*$ and $z_1, z_2 \in S^2$. Let $\varphi_{\theta,z}\ \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ be a rotation around the axis $z \in \mathbb{C}P^1$ by the angle $\theta$. **Consider the diffeomorphism $\Psi: M \longrightarrow M$ mapping $(\theta_1, \theta_2, z_1, z_2)$ to $(\theta_1, \theta_2, z_1, \varphi_{\theta_1, z_1}(z_2))$.**

**THEOREM:** Let $\omega_\lambda$ be the product symplectic form on $M = T^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ obtained as a product of symplectic forms of volume 1, 1, $\lambda$ on $T^2$, $\mathbb{C}P^1$, $\mathbb{C}P^1$. **The form $\Psi^*(\omega_1)$ is homologous, but not diffeomorphic to $\omega_1$.** However, **the form $\Psi^*(\omega_\lambda)$ is diffeomorphic to $\omega_\lambda$ for any $\lambda \neq 1$.**

(D. McDuff, *Examples of symplectic structures*, Invent. Math. 89 (1987), 13-36.)

## Space of symplectic packings

**DEFINITION:** Let $\mathfrak{P}$ be the space of all symplectic packing of $n$ balls in $M$, equipped with a natural Frechet manifold structure (with topology of uniform convergence of derivatives). There is an action of $\operatorname{Symp}(B)^n \times \operatorname{Symp}(M)$, where $\operatorname{Symp}(B)$ is the group of symplectomorphisms of a ball. Denote by $\operatorname{Symp}_0(B)$, $\operatorname{Symp}_0(M)$ their connected components (groups of **symplectic isotopies**). **Teichmüller space of symplectic packings** $\operatorname{Teich}_{ba}$ is $\frac{\mathfrak{P}}{\operatorname{Symp}_0(B)^n \times \operatorname{Symp}_0(M)}$.

**LEMMA:** Let $\varphi_1, \varphi_2 : B \longrightarrow M$ be two balls of the same radius in $M$. **Then $\varphi_1, \varphi_2$ correspond to the same point of $\operatorname{Teich}_{ba}$,** for $\varphi_1, \varphi_2$ sufficiently close.

**Proof:** Consider symplectic structure $\omega_M - \omega_B$ on $M \times B$. The map $\varphi_i$ is a symplectomorphism if and only if its graph in $M \times B$ is Lagrangian. However, in a neighbourhood of a Lagrangian subvariety $L \subset N$, symplectic manifold is symplectomorphic to $T^*L$ (Weinstein), and Lagrangian subvarieties of $T^*L$ which project bijectively to $L$ are given by closed forms on $L$. Finally, all exact forms are related by a Hamiltonian diffeomorphism of $T^*L$. ■

**COROLLARY:** Consider a map $\operatorname{Per}: \operatorname{Teich}_{ba} \longrightarrow \mathbb{R}^n$, mapping a packing of radii $r_1, ..., r_n$ to $(r_1, ..., r_n) \in \mathbb{R}^n$. **Then $\operatorname{Per}$ is locally a diffeomprhism.** ■

## Space of symplectic packings in $\mathbb{R}^n$.

**THEOREM: ("Alexander's trick")** Let $\varphi_1 : B \longrightarrow M$ be two balls of the same radius in $\mathbb{R}^{2n}$. **Then $\varphi_1, \varphi_2$ correspond to the same point of $\operatorname{Teich}_{ba}$.**

**Proof. Step 1:** Using parallel transport and a linear map, we may assume that $d\varphi_1|_0 = d\varphi_2|_0$. From the above argument it follows that there exists a Hamiltonian isotopy mapping $\varphi_1$ to $\varphi_2$ in a small neighbourhood of 0. Therefore, we may assume that $\varphi_1 = \varphi_2$ in a small naighbourhood of 0.

**Step 2:** Assume that $\varphi_1$ is the standard ball. Take the map $\psi_t := t\varphi_2(x/t)$. For small values of $t$ it is the standard ball $\varphi_1$, for $t = 1$ it is $\varphi_2$. We constructed an isotopy from $\varphi_1$ to $\varphi_2$. ■

**COROLLARY:** Let $\operatorname{Teich}_{ba}(\mathbb{R}^{2k})$ be the Teichmüller space of 1 ball in $\mathbb{R}^{2k}$, and $\operatorname{Per}: \operatorname{Teich}_{ba} \longrightarrow \mathbb{R}$ the period map. **Then $\operatorname{Per}$ is a diffeomorphism.** ■

## Star-shaped manifolds.

**DEFINITION:** Let $H$ be a multiplicative semigroup of real numbers in $]0,1]$, and $H^n$ its $n$-th power. We say that $H^n$ **acts semi-tramsitively** on a manifold $M$ if for all points $x, y \in M$ there exist $h_1, h_2 \in H^n$ such that $h_1(x) = h_2(y)$. **Star-shaped manifold** is a manifold (not necessarily Hausdorff) with locally free, semi-transitive action of $H^n$.

**EXAMPLE:** Any interval $]0,k[$ with action of $H$ putting $\lambda, z$ to $\lambda z$ is star-shaped.

**EXAMPLE:** Let $a, b, c$ be positive numbers, with $c < a, b$, and let $M$ be a non-Hausdorff manifold obtained from $]0,a[$ and $]0,b[$ by gluing an interval $]0,c[\subset]0,a[$ to $]0,c[\subset]0,b[$. **Then $M$ is a star-shaped manifold,** with the action of $H$ as above.

**CLAIM: All 1-dimensional star-shaped manifolds are given by rooted metric trees,** with orientation facing away from the origin (designated as 0), glued together as above.

## $\operatorname{Teich}_{ba}$ is star-shaped

Let $M$ be a symplectic manifold, and $\operatorname{Teich}_{ba}$ the the Teichmüller space of $n$ symplectic balls in $M$. Consider the action of $(\lambda_1, ..., \lambda_n) \in H^n$ which replaces the balls of radii $r_1, ..., r_n$ by the balls of radii $\lambda_1 r_1, ..., \lambda_n r_n$ replacing the map $\varphi_i : B_{r_i} \longrightarrow M$ by its restriction to $B_{\lambda_i r_i} \subset B_{r_i}$.

**THEOREM: This action defines the structure of star-shaped manifold on $\operatorname{Teich}_{ba}$.**

**Proof. Step 1:** The action $\rho$ of $H^n$ is locally free, because $\operatorname{Teich}_{ba}$ is equipped with an etale projection to $(\mathbb{R}^{>0})^n$, and **this projection commutes with the star-shaped structure on $(\mathbb{R}^{>0})^n$.**

**Step 2:** It remains to show that the $H^n$-action is semitransitive. Let's start by making a symplectomorphism putting centers of balls in the packing $\mathfrak{x}$ to centers of balls in $\mathfrak{y}$. Prove that the group of symplectomprhisms acts transitively on $n$-tuples of distinct points on $M$ **(an exercise).**

**Step 3:** Sufficiently small balls in $M$ of the same radius are symplectically isotopic, because they are isotopic in $\mathbb{R}^{2k}$, hence for $\lambda_1, ..., \lambda_n, \lambda'_1, ..., \lambda'_n$ small enough, $\rho(\lambda_1, ..., \lambda_n)(\mathfrak{x}) = \rho(\lambda'_1, ..., \lambda'_n)(\mathfrak{y})$ for any points $\mathfrak{x}$ and $\mathfrak{y} \in \operatorname{Teich}_{ba}$ such that the corresponding balls have radii $r_1, ..., r_n$ and $\frac{r_1\lambda_1}{\lambda'_1}, ..., \frac{r_n\lambda_n}{\lambda'_n}$. ■

## Hausdorff star-shaped manifolds

**CLAIM:** Let $M$ be a Hausdorff star-shaped manifold, equipped with an $H^n$-equivariant map $\Psi$ to $(\mathbb{R}^{>0})^n$. **Then $\Psi$ is an open embedding.**

**Proof:** Clearly, $\Psi$ is locally a diffeomorphism. Suppose that $\Psi$ is not injective, and let $\Psi(x) = \Psi(y)$. Then, for some $\lambda \in H^n$, one has $\lambda(x) = \lambda(y)$. Consider $\lambda$ as a point on one-parametric semigroup $\lambda_t := te^{\log \lambda}$, $t \in ]0,1]$. Let $t_0$ be the supremum of all $t$ such that $\lambda_t(x) = \lambda_t(y)$. Since $\lambda_t$ is locally a diffeomorphism, $\lambda_{t_0}(x) = \lambda_{t_0}(y)$ implies that $\lambda_{t_0+\varepsilon}(x) = \lambda_{t_0+\varepsilon}(y)$, for sufficiently small $\varepsilon$. Therefore, $\lambda_{t_0}(x) \neq \lambda_{t_0}(y)$ while $\lambda_{t_0-\varepsilon}(x) \neq \lambda_{t_0-\varepsilon}(y)$ for all $\varepsilon > 0$. Then $M$ is not Hausdorff. ■

**COROLLARY:** Suppose that $\operatorname{Teich}_{ba}$ is Hausdorff. Then all symplectic packing by balls of radius $r_1, ..., r_n$ are isotopic. ■

**REMARK:** The same is true if we take the quotient by all symplectomorphisms instead of symplectic isotopies. **The Hausdorffness of the space of symplectic packing up to symplectomorphisms would imply uniqueness of packing by balls of given radii up to symplectomorphisms.**

## Symplectic quotient

**DEFINITION:** Let $\rho$ be an $S^1$-action on a symplectic manifold $(M,\omega)$ preserving the symplectic structure, and $\vec{v}$ its unit tangent vector. Cartan's formula gives $0 = \operatorname{Lie}_{\vec{v}}\omega = d(\omega \lrcorner \vec{v})$, hence $\omega \lrcorner \vec{v}$ is a closed 1-form. **Hamiltonian**, or **moment map** of $\rho$ is an $S^1$-invariant function $\mu$ such that $d\mu = \omega \lrcorner \vec{v}$, and **symplectic quotient** $M/\!\!/_c S^1$ is $\mu^{-1}(c)/S^1$.

**REMARK:** In these assumptions, restriction of the symplectic form $\omega$ to $\mu^{-1}(c)$ vanishes on $\vec{v}$, hence it is **obtained as a pullback of a closed 2-form $\omega_{/\!\!/}$ on $M/\!\!/_c S^1$.**

**THEOREM: The form $\omega_{/\!\!/}$ is a symplectic form on $M/\!\!/_c S^1$.** In other words, **the symplectic quotient is a symplectic manifold.**

**REMARK:** If, in addition, $M$ is equipped with a Kähler structure $(I,\omega)$, and $S^1$-action preserves the complex structure, the symplectic quotient $M/\!\!/_c S^1$ inherits the Kähler structure. In this case it is called **a Kähler quotient**. Whenever the $S^1$-action can be integrated to holomorphic $\mathbb{C}^*$-action, **the Kähler quotient is identified with an open subset of its orbit space.**

**REMARK:** The moment map is defined by $d\mu = \omega \lrcorner \vec{v}$ uniquely up to a constant. However, the symplectic quotient $M/\!\!/_c S^1 = \mu^{-1}(c)/S^1$ **depends heavily on the choice of $c \in \mathbb{R}$.**

## Symplectic blow-up

**CLAIM:** Consider the standard $S^1$-action on $\mathbb{C}^n$, and let $W \subset \mathbb{C}^n$ be an $S^1$-invariant open subset. Consider the product $V := W \times \mathbb{C}$ with the standard symplectic structure and take the $S^1$-action on $\mathbb{C}$ opposite to the standard one. **Then its moment map is $w - t$, where $w(x) = |x|^2$ is the length function on $W$ and $r(t) = |t|^2$ the length function on $\mathbb{C}$.**

**DEFINITION:** **Symplectic cut** of $W$ is $(W \times \mathbb{C}) /\!\!/_c S^1$.

**REMARK:** Geometrically, the symplectic cut is obtained as follows. Take $c \in \mathbb{R}$, and let $W_c := \{w \in W \ \ | \ \ |w|^2 \leqslant c\}$. Then $W_c$ is a manifold with boundary $\partial W_c$, which is a sphere $|w|^2 = c$. Then $(W \times \mathbb{C}) /\!\!/_c S^1 = (W_c \times \mathbb{C}) /\!\!/_c S^1$ is obtained from $W_c$ by gluing each $S^1$-orbit which lies on $\partial W_c$ to a point. Combinatorially, $(W \times \mathbb{C}) /\!\!/_c S^1$ is $\mathbb{C}^n$ with 0 replaced with $\mathbb{C}P^{n-1}$.

**DEFINITION:** In these assumptions, **symplectic blow-up** of radius $\lambda = \sqrt{c}$ of $W$ in 0 is $(W \times \mathbb{C}) /\!\!/_c S^1$. **Symplectic blow-up** of a symplectic manifold $M$ is obtained by removing a symplectic ball $W$ and gluing back a blown-up symplectic ball $(W \times \mathbb{C}) /\!\!/_c S^1$.

## Symplectic blow-down

**DEFINITION:** Let $E \subset M$ be a submanifold diffeomorphic to $\mathbb{C}P^{k-1}$. Fix the diffeomorphism $E \xrightarrow{\sim} \mathbb{C}P^{k-1}$. We say that a symplectic structure $\omega$ on $M$ **is framed at $\mathbb{C}P^{k-1}$** if $\omega$ it is proportional to the Fubini-Study symplectic form on $E$, and the symplectic normal bundle to $E$ has $c_1(NE) = -[H]$, where $[H]$ is the hyperplane class. The notation $c_1(NE)$ makes sense because $NE$ is an oriented 2-dimensional real vector bundle. We say that $\omega$ is **weakly framed** if $(\omega|_E)^{1,1}$ is Hermitian and $c_1(NE) = -[H]$.

**THEOREM: (Weinstein)** Let $M_1 \subset M$ be a symplectic submanifold. Then there exists a neighbourhood of $M_1$ in $M$ which is symplectomorphic to a neighbourhood of the zero section in the total space of the normal bundle $NM_1$ equipped with the standard symplectic structure making $NM_1$ locally a product of $M_1$ and the fibers of the projection $NM_1 \longrightarrow M_1$.

## Symplectic blow-down (2)

**REMARK:** Using the condition $c_1(NE) = -[H]$ and Weinstein's theorem, we obtain immediately that **a small neighbourhood of $E$ is symplectomorphic to a small neighbourhood of a blow-up of a point in a complex space.**

**DEFINITION:** Let $E_1, ..., E_n$ be the set of submanifolds in a $2n$-dimensional manifold, diffeomorphic to $\mathbb{C}P^{k-1}$, and $\omega$ a symplectic structure framed in $E_1, ..., E_n$. Remove a neighbourhood of each $E_i$ such that a neighbourhood of its boundary is symplectomorphic to a neighbourhood of a sphere in $\mathbb{R}^{2k}$, and glue a symplectic ball inside. We obtain a symplectic manifold, equipped with a symplectic packing, which is called **symplectic blow-down of $(M, E_1, ..., E_n)$.**

## Framed Teichmüller space

**DEFINITION:** Let $E_1, ..., E_n$ be the set of submanifolds in a $2n$-dimensional manifold $\tilde{M}$, diffeomorphic to $\mathbb{C}P^{k-1}$, and $\operatorname{Teich}_{\mathrm{fr}}$ the **Teichmüller space of weakly framed symplectic structures,** that is, the quotient of the set of symplectic structures by the set of isotopies acting trivially on $E_i$. Consider the symplectic blow-down $M$ of $\tilde{M}$ with its symplectic form $\omega$. This gives a natural map $B: \operatorname{Teich}_{\mathrm{fr}} \longrightarrow \operatorname{Teich}(M)$, where $\operatorname{Teich}(M)$ is the Teichmüller space of symplectic structures on $M$, called **the blow-down map**.

**REMARK:** The map $B$ is a smooth submersion. Indeed, the symplectic Teichmüller spaces are equipped with the charts given by cohomology, and on cohomology $B$ is just a forgetful map.

**Exercise:** Using Weinstein theorem and Moser, prove that **the space** $\operatorname{Teich}_{\mathrm{fr}}$ **with weak framing is equal to** $\operatorname{Teich}_{\mathrm{fr}}$ **with framing.**

## McDuff-Polterovich Teichmüller space

**REMARK:** The map $B : \operatorname{Teich}_{\mathsf{fr}} \longrightarrow \operatorname{Teich}(M)$ has partial inverse: given a set of open balls in $M$ which lie in a non-intersecting Darboux charts, **we can always use the symplectic cut construction giving the symplectic blow-up of $M$**, with $n$ exceptional divisors corresponding to the $n$ balls.

**DEFINITION: The McDuff-Polterovich Teichmüller space of blow-ups** $\operatorname{Teich}_{bu}$ associated with $(M, \omega)$ is $\Psi^{-1}(\omega) \subset \operatorname{Teich}_{\mathsf{fr}}$.

**REMARK:** The symplectic blow-down map maps a (weakly) framed symplectic structure $\tilde\omega$ on $\tilde M$ to a set of symplectic balls on $M$. This defines a map $MP : \operatorname{Teich}_{bu} \longrightarrow \operatorname{Teich}_{ba}$.

**THEOREM: (McDuff-Polterovich) The map $MP : \operatorname{Teich}_{bu} \longrightarrow \operatorname{Teich}_{ba}$ is a diffeomorphism.**

**Proof:** To produce the inverse, we take a symplectic packing, and glue the fibers of the Hopf map on a boundary of each ball. After gluing these circles, the boundary of a ball becomes $E = \mathbb{C}P^{k-1}$, with weak framing, and the symplectic structure naturally restricts to $E$ giving a symplectic structure here. ■

## McDuff-Polterovich construction for Kähler manifolds

**DEFINITION:** Let $(M, \omega)$ be a Kähler manifold, and $\tilde{M} \xrightarrow{\pi} M$ its blow-up in $n$ points. Define $\operatorname{Teich}_{bu}^K$ as the space of all Kähler classes on $\tilde{M}$ of form $\pi^*\omega - \sum c_i E_i$. We call this space **the Teichmüller space of Kähler blow-ups**.

**REMARK:** The period map $\operatorname{Per}: \operatorname{Teich}_{bu}^K \longrightarrow H^2(M)$ is injective. Indeed, any two Kähler forms with the same cohomology class are connected by a path $t\omega + (1-t)\omega'$ consisting of forms in the same cohomology class. In particular, $\operatorname{Teich}_{bu}^K$ **is a priori Hausdorff**.

**REMARK:** Since $\operatorname{Teich}_{bu}$ is connected, **to show that it is Hausdorff it would suffice to prove that** $\operatorname{Teich}_{bu}^K = \operatorname{Teich}_{bu}$.

## Closure of the Kähler cone

**PROPOSITION:** $(M,\omega)$ be a Kähler manifold, $\tilde M \xrightarrow{\pi} M$ its blow-up in $n$ points, and $\operatorname{Teich}^k_{bu} \subset \operatorname{Teich}_{bu}$ the Teichmüller spaces constructed above. Suppose that the closure $\overline{\operatorname{Teich}^K_{bu}}$ in $\operatorname{Teich}_{bu}$ is Hausdorff, and $\operatorname{Per}(\operatorname{Teich}^K_{bu}) = \operatorname{Per}(\operatorname{Teich}_{bu})$, where Per maps a class $\pi^*\omega - \sum c_i E_i$ to $c_1, ..., c_n$ (this is the same period map as we considered for $\operatorname{Teich}_{ba}$). **Then $\operatorname{Teich}_{bu}$ is Hausdorff.**

**Proof:** Let $x$ be a point in the closure of $\operatorname{Teich}_{bu} \setminus \operatorname{Teich}^K_{bu}$ contained in $\operatorname{Teich}^K_{bu}$; such a point exists because $\operatorname{Teich}_{bu}$ is connected. Assume that $x = \lim x_i$, where $x_i \in \operatorname{Teich}^K_{bu}$. Then $\operatorname{Per}(x) = \lim \operatorname{Per}(x_i)$. Since $\operatorname{Per}\Big|_{\operatorname{Teich}^K_{bu}}$ is a diffeomorphism, this implies that $x_i$ converge to a point $x' \in \operatorname{Teich}^K_{bu}$ with the same periods. Then $x \in \overline{\operatorname{Teich}^K_{bu}}$, and this space is non-Hausdorff. ■

## CSC Kähler metrics

**DEFINITION: CSC Kähler metric** is a Kähler metric of constant scalar curvature.

**THEOREM: (Chen-Tian)** CSC Kähler metric is unique, up to a holomorphic automorphism, in each Kähler class.

**THEOREM:** Let $\tilde{M}$ be a Kähler manifold, and $K_1 \subset \operatorname{Kah}(M)$ a closed subset of its Kähler cone. Assume that $M$ has compact group of complex automorphisms, and for each $\eta \in \operatorname{Kah}(M)$, $\eta$ can be represented by the (a posteriori, unique) CSC metric. **Then the closure of $K^\perp$ in symplectic Teichmüller space is Hausdorff.**

**Proof:** Let $\eta_i$ be a sequence of Kähler forms in $K_1$ converging to a symplectic form $\eta$, and $\eta_i'$ the corresponding CSC Kähler forms. Since the map $\eta_i \longrightarrow \eta_i'$ is continuous, $\eta_i'$ converge to a CSC Kähler form, and the limit is unique, because CSC forms uniformly depend on the Kähler class. ■

**COROLLARY:** Let $\tilde{M}$ be a blow-up of a generic deformation of a hyperkähler manifold $(M, \omega)$ or a torus, and $K_1$ the set of Kähler classes $\pi^*\omega - \sum c_i E_i$. **If each $\eta \in K_1$ can be represented by a CSC form, then $\operatorname{Teich}_{ba}$ is Hausdorff.**