

# **Kähler threefolds without subvarieties**

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**(joint work with F. Campana and J.-P. Demailly)**

## Campana simple manifolds

**REMARK:** Recall that **an algebraic dimension**  $a(M)$  of a compact complex manifold  $M$  is the transcendence degree of the field of global meromorphic functions on  $M$ . It is known to be bounded by  $\dim M$ , When  $a(M) = \dim M$ ,  $M$  is called **Moishezon**. Moishezon manifolds are bimeromorphically equivalent to projective ones.

**CLAIM:** **Kähler Moishezon manifolds are projective** (Moishezon).

**DEFINITION:** **Campana simple manifold** is a manifold which is not a union of its proper complex subvarieties.

**REMARK:** For Campana simple manifolds, **algebraic dimension is 0**.

**EXAMPLE:** General deformation of a complex torus of dimension  $\geq 2$  has no subvarieties.

**EXAMPLE:** **General deformation of a Hilbert scheme of a K3 has no subvarieties** (V., 1997).

**EXAMPLE:** **General deformation of a maximal holonomy hyperkähler manifold is Campana simple** (V., 1996).

## Threefolds without subvarieties

**CONJECTURE:** (Campana)

**Any Campana simple threefold is bimeromorphic to a torus.** Any Campana simple manifold is bimeromorphic to a torus or a hyperkähler orbifold.

**Theorem 1:** (Campana-Demailly-V.)

Let  $M$  be a compact Kähler 3-dimensional manifold. Assume that  $M$  has no non-trivial closed complex subvarieties. **Then  $M$  is a complex torus.**

## Calabi-Yau manifolds

**DEFINITION: A Calabi-Yau manifold** is a compact Kaehler manifold with topologically trivial canonical bundle.

### THEOREM: (Calabi-Yau)

Let  $(M, I, g)$  be Calabi-Yau manifold. **Then there exists a unique Ricci-flat Kaehler metric in any given Kaehler class.**

**THEOREM: (Bogomolov's decomposition)** Let  $M$  be a compact, Ricci-flat Kaehler manifold. **Then there exists a finite covering  $\tilde{M}$  of  $M$  which is a product of Kaehler manifolds of the following form:**

$$\tilde{M} = T \times M_1 \times \dots \times M_i \times K_1 \times \dots \times K_j,$$

with all  $M_i, K_i$  simply connected,  $T$  a torus, and  $\mathcal{H}ol(M_l) = Sp(n_l)$ ,  $\mathcal{H}ol(K_l) = SU(m_l)$

**COROLLARY:** Any Campana simple Calabi-Yau manifold **is either a torus or a simple hyperkähler manifold.**

**REMARK:** Theorem 1 follows immediately if we prove that the canonical bundle  $\Lambda^{3,0}(M)$  is trivial, because there are no hyperkähler manifolds in dimension 3.

## Subvarieties in a torus

**THEOREM:** Let  $Z$  be a subvariety of a torus  $T$ . Then **the canonical bundle of  $Z$  is base point free and globally generated** (that is, semiample). Moreover, **the corresponding map  $\varphi : Z \rightarrow \mathbb{P}H^0(Z, K_Z)^*$  is a non-trivial holomorphic map, unless  $Z$  is a subtorus of  $T$ .**

**Proof. Step 1:** The bundle  $\Omega^d T$ ,  $d = \dim Z$  is globally generated, hence restrictions of the sections of  $\Omega^d T$  to  $Z$  don't have base points (common zeros), and produce a holomorphic map to  $\mathbb{C}P^m$ .

**Step 2:** The map  $\varphi$  is non-trivial unless the  $K_Z$  is a trivial bundle. In the latter case, the rank of restriction map  $H^0(\Omega^d T) \rightarrow H^0(Z, K_Z)$  is 1, that is, **all tangent spaces to  $Z$  are parallel in  $T$** , and  $Z$  is a torus. ■

## Albanese map for Campana simple manifolds

**COROLLARY:** Let  $M$  be a Campana simple manifold, such that  $H^1(M) \neq 0$ . **Then  $M$  is a torus.**

**Proof:** Consider the Albanese map  $M \rightarrow H^{1,0}(M)/H^1(M, \mathbb{Z})$ . It is non-trivial; **since  $M$  is not covered by proper subvarieties,  $M$  is embedded to a torus  $T$ .** Then  $M$  is a subtorus of  $T$ , because the holomorphic map  $\varphi: M \rightarrow \mathbb{P}H^0(M, K_M)^*$  has to be trivial. ■

**REMARK:** To prove that a Campana simple threefold  $M$  is a torus, **it would suffice either to show that  $K_M = \mathcal{O}_M$  or to show that  $H^1(M, \mathbb{R}) \neq 0$ .**

## Brunella's theorem

**DEFINITION:** A line bundle is called **pseudoeffective** if it admits a singular metric with positive curvature current.

**DEFINITION:** A **1-dimensional holomorphic foliation** on a complex manifold is a coherent subsheaf  $\mathcal{F} \subset TM$  of rank 1.

**DEFINITION:** A manifold  $M$  is called **uniruled** if it is a union of rational curves  $C_z \subset M$ .

**DEFINITION:** Let  $F$  be a coherent sheaf on a complex manifold, and  $F^{**}$  its double dual. Then  $F^{**}$  is called **reflexization** of  $F$ .

**CLAIM:** Reflexization of a rank one sheaf is always a line bundle.

**THEOREM:** (M. Brunella)

Let  $M$  be a compact Kähler manifold equipped with a 1-dimensional holomorphic foliation  $\mathcal{F} \subset TM$ , and  $F := \mathcal{F}^{**}$  its reflexization. **Then either  $F^*$  is pseudoeffective, or  $M$  is uniruled.**

## Brunella theorem for Campana simple threefolds

**THEOREM:** (Kodaira) Let  $M$  be a Kähler manifold, and  $[\omega] \in H^2(M, \mathbb{R})$  its Kähler class. Suppose that  $\omega$  is rational. **Then  $M$  is projective.**

**COROLLARY:** Let  $M$  be a non-projective compact Kähler manifold. **Then  $H^{2,0}(M) \neq 0$ ,** in other words,  $M$  admits a holomorphic 2-form.

**Proof:** Assume that  $H^{2,0}(M) = 0$ . Then  $H^{1,1}(M) = H^2(M)$ . Since the Kähler cone is open in  $H^{1,1}(M, \mathbb{R})$ , it contains a rational point, hence  $M$  is projective. ■

**THEOREM:** Let  $M$  be a Campana simple 3-dimensional compact Kähler manifold. **Then its canonical bundle  $K_M$  is pseudoeffective.**

**Proof:** Consider a non-zero holomorphic 2-form  $\Omega \in \Omega^2 M$ , and let  $\mathcal{F}$  be the kernel of  $\Omega$ . Since the rank of  $\Omega$  is 2, one has  $\text{rk } \mathcal{F} = 1$ . **Brunella's theorem implies that  $F^*$  is pseudoeffective,** because otherwise  $M$  is uniruled. However, the normal bundle  $TM/\mathcal{F}$  is equipped with a non-degenerate 2-form  $\Omega$ , hence its determinant vanishes. This gives  $F^* = K_M$ . ■

## Big and nef currents

**DEFINITION:** Let  $M$  be a compact,  $n$ -dimensional complex manifold. A positive, closed  $(1,1)$ -current  $\theta$  is called **nef** if it is a limit of positive, closed  $(1,1)$ -forms. If the cohomology class  $[\theta]$  in addition satisfies  $\int_M [\theta]^n > 0$ , then  $\theta$  is called **big**.

**THEOREM:** Let  $L$  be a holomorphic line bundle on a Kähler manifold with  $c_1(L)$  represented by a big and nef class (such a bundle is called **big and nef**). **Then the function  $P(N) := \dim H^0(L^N)$  grows as a polynomial of degree  $\dim M$ .**

**Proof:** This result follows from the Demailly's holomorphic Morse inequalities.

■

**REMARK:** Such a rate of growth guarantees that  $M$  is Moishezon. Then, **a Kähler manifold admitting a big and nef class is projective.**

**COROLLARY:** Therefore, **a Campana simple threefold has no big and nef bundles.**

## Lelong numbers and Lelong sets

**DEFINITION:** Let  $T$  be a positive  $(p, p)$ -current on an  $n$ -dimensional manifold  $M$ ,  $z \in M$  a point, and  $d_z$  distance to  $z$ . Consider the current  $dd^c \log d_z$ . In a neighbourhood of  $z$  it is positive, and one can take the product  $T \wedge (dd^c \log d_z)^{n-p}$ . Its mass at  $z$  is called **the Lelong number of  $T$  at  $z$** , denoted by  $\nu_z(T)$ . It measures how singular  $T$  is at  $Z$ .

**DEFINITION:** Let  $c > 0$  be a number. **Lelong set**  $F_c$  of  $T$  is the set of  $z$  such that  $\nu_z(T) \geq c$ .

**THEOREM:** (Siu) **Lelong sets are complex analytic subvarieties of  $M$ .**

**THEOREM:** (Nadel) Let  $h$  be a singular metric on a line bundle  $L$ . Consider the sheaf  $L_{L^2}$  of  $L^2$ -integrable holomorphic sections of  $L$ . **Then  $L_{L^2}$  is a coherent subsheaf of  $L$ .**

**DEFINITION:** The **multiplier ideal** of  $h$  is an ideal  $I_h$  such that  $I_h \otimes L = L_{L^2}$ .

**THEOREM:** (Nadel) **Support of a multiplier ideal of a bundle  $L$  is a Lelong set of its curvature current.**

## Demailly's regularization of currents

**THEOREM:** Any positive, closed (1,1)-current  $T$  on a Kähler manifold  $(M, \omega)$  can be approximated by a sequence  $T_k$  of closed, real (1,1)-currents in the same cohomology class satisfying the following:

(A)  $T_k + \delta_k \omega \geq 0$ , where  $\{\delta_k\} \rightarrow 0$ .

(B)  $T_k$  are smooth outside of a complex analytic subset  $Z_k \subset M$ , with  $Z_1 \subset Z_2 \subset \dots$

(C) Let  $T_0$  be a smooth form cohomologous to  $T$ . Then  $T_k = T_0 + dd^c \psi_k$ , where  $\psi_k$  is a non-increasing sequence of almost plurisubharmonic functions converging to an almost plurisubharmonic  $\psi$ , which satisfies  $dd^c \psi + T_0 = T$ .

(D) Locally around  $Z_k$ , the functions  $\psi_k$  have logarithmic poles:

$$\psi_k = \lambda_k \log \sum |g_{k,l}|^2 + \tau_k,$$

where  $g_{k,l}$  are holomorphic functions vanishing on  $Z_k$ , and  $\tau_k$  is smooth.

(E) The Lelong numbers  $\nu(T_k, x)$  of  $T_k$  are non-decreasing in  $k$  for any  $x \in M$  and converge to  $\nu(T, x)$ .

**REMARK:** In layman's terms, **any positive, closed (1,1)-current can be approximated by a sequence of positive currents with logarithmic singularities** and their Lelong sets would also converge.

**COROLLARY:** In particular, **any current with empty Lelong sets is nef.**

## Currents with isolated singularities

**THEOREM:** Let  $\theta$  be a positive  $(1,1)$ -current on a complex  $n$ -manifold, with all Lelong sets of dimension 0. **Then  $\theta$  is nef. If, in addition, its Lelong sets are non-empty, then  $\theta$  is also big.**

**Proof. Step 1:** The nef condition follows directly from approximation. Indeed,  $\theta$  is locally approximated by a sum of a smooth form  $\theta_k$  and a  $dd^c$ -exact current  $dd^c\psi_k$  with isolated logarithmic singularities which locally look like  $dd^c \log |x_0 - x|$ . Then  $\theta_k + \max(\psi_k, -C_k)$  is non-singular and approximates  $\theta$ , for an appropriate sequence  $C_k \rightarrow \infty$ .

**Step 2:** The top product  $(dd^c \log |x_0 - x|)^n$  is a positive volume form, giving  $\int_M \theta^n > 0$ . ■

## Positive currents on 3-manifolds without subvarieties

**COROLLARY:** Let  $M$  be a Kähler manifold without non-trivial subvarieties. Then any positive, closed (1,1)-current  $\theta$  on  $M$  with integer cohomology class has empty Lelong sets.

**Proof:** As shown above,  $\theta$  is nef, has isolated Lelong sets, and is big unless they are empty. Then a holomorphic line bundle  $L$  with  $c_1(L) = [\theta]$  is big and nef, making  $M$  projective. ■

**COROLLARY:** Let  $M$  be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties. Then the canonical bundle  $K_M$  is nef, not big, and admits a singular metric with all Lelong sets empty.

**Proof:** By Brunella's theorem,  $K_M$  admits a singular metric with positive curvature current  $\theta$ . Then  $\theta$  is nef and has empty Lelong sets as shown above. ■

## Demailly-Peternell-Schneider vanishing theorem

The following theorem was rediscovered several times during 1990-ies (Enoki, Takegoshi, Morougane). Its most general form (which we use) is due to Demailly, Peternell and Schneider.

**THEOREM:** Let  $(M, I, \omega)$  be a compact Kähler manifold,  $\dim_{\mathbb{C}} M = n$ ,  $K$  its canonical bundle, and  $L$  a holomorphic line bundle on  $M$  equipped with a singular Hermitian metric  $h$ . Assume that the curvature  $\Theta$  of  $L$  is a positive current on  $M$ , and denote by  $\mathcal{I}(h)$  the corresponding multiplier ideal. Then **the wedge multiplication operator  $\eta \rightarrow \omega^i \wedge \eta$  induces a surjective map**

$$H^0(\Omega^{n-i} M \otimes L \otimes \mathcal{I}(h)) \xrightarrow{\omega^i \wedge \cdot} H^i(K \otimes L \otimes \mathcal{I}(h)).$$

Here  $\omega$  is considered as an element in  $H^1(\Omega^1 M)$ , and multiplication by  $\omega$  maps  $H^k(\Omega^{n-l} M \otimes F)$  to  $H^{k+1}(\Omega^{n-l+1} M \otimes F)$ . ■

## Demailly-Peternell-Schneider for threefolds without subvarieties

Demailly-Peternell-Schneider, applied to 3-manifolds without subvarieties, gives the following vanishing result.

**COROLLARY:** Let  $M$  be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties, and  $K$  its canonical bundle.

**Then the multiplication map**

$$H^0(\Omega^{n-i}M \otimes K^m) \xrightarrow{\omega^i \wedge \cdot} H^i(K^{m+1})$$

**is surjective for any  $m \geq 0$ .**

**Proof:** The bundle  $K^m$  is pseudoeffective, and all its Lelong numbers vanish. Then the multiplier ideals for  $K^m$  are trivial. ■

## Bounds on Euler characteristic

**CLAIM:** Let  $M$  be a complex manifold without divisors, and  $B$  a holomorphic bundle on  $M$ . **Then**  $\dim H^0(B) \leq \text{rk } B$ .

**Proof:** Otherwise there would be a non-constant linear relation between sections; being non-constant, it must vanish on a divisor. ■

**PROPOSITION:** Let  $M$  be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties,  $K$  its canonical bundle, and  $A(m) := \chi(K^m)$  the holomorphic Euler characteristic of the pluricanonical bundle. **Then**  $|A(m)| \leq 4$  for any  $m \geq 1$ .

**Proof:** From Demailly-Peternell-Schneider it follows that

$$\begin{aligned} -\dim H^0(\Omega^1 M \otimes K^{m-1}) - \dim H^0(\Omega^3 M \otimes K^{m-1}) &\leq \\ &\leq \chi(K^m) \leq \dim H^0(\Omega^0 M \otimes K^{m-1}) + \dim H^0(\Omega^2 M \otimes K^{m-1}). \end{aligned}$$

The corresponding bundles have  $\text{rk} \leq 4$ . By the above statement, their spaces of sections are no more than 4-dimensional. ■

## Riemann-Roch-Hirzebruch and its applications

**THEOREM:** (Riemann-Roch-Hirzebruch)

Let  $\chi(M, E) := \sum_{i=0}^{\dim_{\mathbb{C}} M} (-1)^i \dim_{\mathbb{C}} H^i(M, E)$  be a holomorphic Euler characteristic of  $E$ . **Then for any complex threefold  $M$ , one has**

$$\chi(M, K_M^d) = \frac{d+3d^3+2d^3}{12} c_1(K_M)^3 + \frac{(1-12d)}{24} c_1(M)c_2(M). \blacksquare$$

**COROLLARY:** Let  $M$  be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties,  $K_M$  its canonical bundle, and  $A(m) := \chi(M, K_M^m)$ . **Then  $A(m) = \text{const}$ .**

**Proof:**  $A(m)$  is a polynomial by Riemann-Roch-Hirzebruch theorem, but it's bounded, as shown above, hence constant.  $\blacksquare$

**COROLLARY:** Let  $M$  be a 3-dimensional compact Kähler manifold without non-trivial compact complex subvarieties. **Then  $c_1(K_M)^3 = 0$ , and  $c_1(M)c_2(M) = 0$ .**

**Proof:**  $c_1(K_M)^3 = 0$ , because  $K_M$  is nef and not big. Then Riemann-Roch-Hirzebruch formula gives  $A(m) := \frac{(1-12m)}{24} c_1(M)c_2(M)$ . Since  $A(m)$  is bounded, this implies  $c_1(M)c_2(M) = 0$ , giving  $A(m) = 0$ .  $\blacksquare$

## Proof of Theorem 1

The main result (Theorem 1) follows if we prove that  $M$  is Calabi-Yau or  $H^1(M, \mathbb{R}) \neq 0$ .

**COROLLARY:** Let  $M$  be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties. **Then  $K_M$  is trivial.**

**Proof:** From Riemann-Roch-Hirzebruch, we obtain  $\chi(O) = 1 - h^1(O) + h^2(O) - h^0(K_M) = 0$ . Since  $M$  admits a holomorphic 2-form,  $h^2(O) \geq 1$ , which gives  $h^0(K_M) \geq 1$ . **However, a section of  $K_M$  cannot vanish, because  $M$  has no divisors.** This implies that  $K_M$  is trivial. ■