Kahler threefolds without subvarieties

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Differential Geometry seminar,
IMPA, May 10, 2022

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Plan.

1. Introduction: Campana simple manifolds. Main result.

2. Calabi-Yau manifolds.

3. Positive currents.

4. Brunella's pseudoeffectivity for foliations.

5. Demailly-Peternell-Schneider vanishing theorem.

6. Riemann-Roch-Hirzebruch and its applications.

Campana simple manifolds

REMARK: Recall that **an algebraic dimension** $a(M)$ of a compact complex manifold M is the transcendence degree of the field of global meromorphic functions on M. It is known to be bounded by $\dim M$, When $a(M) = \dim M$, M is called **Moishezon**. Moishezon manifolds are bimeromorphically equivalent to projective ones.

DEFINITION: **Campana simple manifold** is a manifold which is not a union of its proper complex subvarieties.

CONJECTURE: (Campana)
Any Campana simple threefold is bimeromorphic to a torus. Any Campana simple manifold is bimeromorphic to a torus or a hyperkähler orbifold.

EXAMPLE: General deformation of a complex torus of dimension $\geqslant 2$ has no subvarieties.

EXAMPLE: General deformation of a Hilbert scheme of a K3 has no subvarieties.

EXAMPLE: General deformation of a hyperkähler manifold is Campana simple.

Threefolds without subvarieties

Theorem 1: (Campana-Demailly-V.)
Let M be a compact Kähler 3-dimensional manifold. Assume that M has no non-trivial closed complex subvarieties. (This automatically implies that M is non-algebraic, and its algebraic dimension is 0.) **Then M is a complex torus.**

Jean-Pierre Demailly (1957-2022)

2010, MFO, photo by Renate Schmid

Calabi-Yau manifolds

DEFINITION: Let M be a complex manifold, $\dim_{\mathbb{C}} M = n$. The holomorphic line bundle $(\Lambda^{n,0}(M), \overline{\partial})$ is called **canonical bundle** of M.

DEFINITION: A Calabi-Yau manifold is a compact Kaehler manifold with topologically trivial canonical bundle.

DEFINITION: Let (M, I, ω) be a Kaehler n-manifold, and $K(M) := \Lambda^{n,0}(M)$ its **canonical bundle.** We consider $K(M)$ as a holomorphic line bundle, $K(M) = \Omega^n M$. The natural Hermitian metric on $K(M)$ is written as $(\alpha, \alpha') \longrightarrow \frac{\alpha \wedge \overline{\alpha}'}{\omega^n}$. Denote by Θ_K the curvature of the Chern connection on $K(M)$. The **Ricci curvature** Ric of M is symmetric 2-form $\mathrm{Ric}(x, y) = \Theta_K(x, Iy)$.

DEFINITION: A Kähler manifold is called **Ricci-flat** if its Ricci curvature vanishes.

THEOREM: (Calabi-Yau)
Let (M, I, g) be Calabi-Yau manifold. **Then each cohomology class of a Kähler form contains a unique Ricci-flat Kaehler form.**

Bogomolov's decomposition theorem

THEOREM: (Bogomolov's decomposition) Let M be a compact, Ricci-flat Kaehler manifold. **Then there exists a finite covering $\tilde{M}$ of M which is a product of Kaehler manifolds of the following form:**

$$\tilde{M} = T \times M_1 \times ... \times M_i \times K_1 \times ... \times K_j,$$

with all M_i, K_i simply connected, T a torus, all M_i are hyperkähler, with $\mathcal{H}\mathrm{ol}(M_i) = Sp(n_i)$, and all K_i Calabi-Yau, $\mathcal{H}\mathrm{ol}(K_i) = SU(m_i)$.

COROLLARY: Any Campana simple Calabi-Yau manifold **is either a torus or a simple hyperkähler manifold.**

REMARK: Theorem 1 follows immediately if we prove that the canonical bundle $\Lambda^{3,0}(M)$ is trivial, because there are no hyperkähler manifolds in dimension 3.

Currents and generalized functions

DEFINITION: Let F be a Hermitian bundle with connection ∇, on a Riemannian manifold M with Levi-Civita connection, and

$$\|f\|_{C^k} := \sup_{x\in M} \Big(|f| + |\nabla f| + ... + |\nabla^k f|\Big)$$

the corresponding **C^k-norm** defined on smooth sections with compact support. **The C^k-topology is independent from the choice of connection and metrics.**

DEFINITION: A generalized function is a functional on top forms with compact support, which is continuous in one of C^i-topologies.

DEFINITION: A k-current is a functional on $(\dim M - k)$-forms with compact support, which is continuous in one of C^i-topologies.

REMARK: Currents are forms with coefficients in generalized functions.

Currents on complex manifolds

DEFINITION: The space of currents is equipped with **weak topology** (a sequence of currents converges if it converges on all forms with compact support). The space of currents with this topology is a **Montel space** (barrelled, locally convex, all bounded subsets are precompact). Montel spaces are **reflexive** (the map to its double dual with strong topology is an isomorphism).

CLAIM: De Rham differential is continuous on currents, and the Poincare lemma holds. Hence, **the cohomology of currents are the same as cohomology of smooth forms.**

DEFINITION: On an complex manifold, **(p,q)-currents** are (p,q)-forms with coefficients in generalized functions

REMARK: In the literature, this is sometimes called $(n-p,n-q)$-currents.

CLAIM: The Poincare and Poincare Dolbeault-Grothendieck lemma hold on (p,q)-currents, and **the d- and $\overline{\partial}$-cohomology are the same as for forms.**

Chern connection

DEFINITION: Let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\overline{\partial}\, B \longrightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition of ∇, $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1}: V \longrightarrow \Lambda^{0,1}(M) \otimes V, \quad \nabla^{1,0}: V \longrightarrow \Lambda^{1,0}(M) \otimes V.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \overline{\partial}$.

DEFINITION: **An Hermitian holomorphic vector bundle** is a smooth complex vector bundle equipped with a Hermitian metric and a holomorphic structure operator $\overline{\partial}$.

DEFINITION: **A Chern connection** on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

Curvature of a holomorphic line bundle

REMARK: If B is a line bundle, $\operatorname{End} B$ is trivial, and **the curvature Θ_B of B is a closed 2-form.**

DEFINITION: Let ∇ be a unitary connection in a line bundle. The cohomology class $c_1(B) := \frac{\sqrt{-1}}{2\pi}[\Theta_B] \in H^2(M)$ is called **the real first Chern class of a line bunlde B**.

REMARK: When speaking of a **"curvature of a holomorphic bundle"**, one usually means the curvature of a Chern connection.

REMARK: Let B be a holomorphic Hermitian line bundle, and b its non-degenerate holomorphic section. Denote by η a (1,0)-form which satisfies $\nabla^{1,0} b = \eta \otimes b$. Then $d|b|^2 = \operatorname{Re} g(\nabla^{1,0} b, b) = \operatorname{Re} \eta |b|^2$. **This gives $\nabla^{1,0} b = \frac{\partial |b|^2}{|b|^2} b = 2\partial \log |b| b$.**

COROLLARY: If $g' = e^{2f} g$ – two metrics on a holomorphic line bundle, Θ, Θ' their curvatures, **one has $\Theta' - \Theta = -2\partial\bar\partial f$**

Positive currents

REMARK: Positive generalized functions are all C^0-continuous as functionals on $C^\infty M$. A positive generalized function multiplied by a positive volume form **gives a measure on a manifold,** and all measures are obtained this way.

DEFINITION: **The cone of positive currents** is generated by $-\sqrt{-1}\,\alpha dz \wedge d\bar z$, where α is a positive generalized function (that is, a measure), and z a holomorphic function.

DEFINITION: Let $Z \subset M$ be a complex analytic subvariety. **The current of integration** $[Z]$ is the current $\alpha \longrightarrow \int_Z \alpha$. **It is closed and positive** (Lelong).

REMARK: (Poincare-Lelong formula) $\frac{\sqrt{-1}}{\pi} dd^c \log|\varphi| = [Z_\varphi]$, where Z_φ is a divisor of a holomorphic function φ.

DEFINITION: A locally integrable function $f: M \longrightarrow [\infty, \infty[$ is called **plurisubharmonic** (psh) if $dd^c f$ is a positive current.

CLAIM: (a local dd^c-lemma) **Locally, every positive, closed (1,1)-current is obtained as $dd^c f$,** for some psh function f.

Singular metrics on line bundles

DEFINITION: Let f be a real locally integrable function on a complex manifold, such that $dd^c f + \alpha$ is a positive current, for some smooth $(1,1)$-form α. Then f is called **almost plurisubharmonic**.

DEFINITION: Let L be a line bundle and h a smooth Hermitian metric on L. For any almost plurisubharmonic function f, we call he^{-f} **a singular metric** on L. Its curvature is equal to $\Theta_h + dd^c f$.

DEFINITION: Let f be an almost plurisubharmonic function, and e^{-f} the corresponding singular metric on a trivial line bundle $\mathcal{O}_M$. **The multiplier ideal** I_f of f is a sheaf of L^2-integrable holomorphic sections of $\mathcal{O}_M$.

THEOREM: (Nadel) **It is a coherent sheaf.**

DEFINITION: **Lelong sets** F_c of an almost plurisubharmonic function f are support varieties for $\mathcal{O}_M / I_{cf}$, where $c \in \mathbb{R}^{>0}$ is a number. **Lelong number** at $x \in M$ is the smallest c such that $F_c \ni x$.

Brunella's theorem

DEFINITION: A line bundle is called **pseudoeffective** if it admits a singular metric with positive curvature current.

DEFINITION: A **holomorphic foliation** on a complex manifold is a coherent subsheaf $\mathcal{F} \subset TM$ which is **involutive**, that is, satisfies $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$.

DEFINITION: A manifold M is called **uniruled** if it is a union of rational curves $C_z \subset M$.

DEFINITION: Let F be a coherent sheaf on a complex manifold, and F^{**} its double dual. Then F^{**} is called **reflexization** of F. It is well known that reflexization of a rank one sheaf is always a line bundle.

THEOREM: (Brunella)
Let M be a compact Kähler manifold equipped with a 1-dimensional holomorphic foliation $\mathcal{F} \subset TM$, and $F := \mathcal{F}^{**}$ its reflexization. **Then either F^* is pseudoeffective, or M is uniruled.**

Kodaira's embedding theorem

THEOREM: Let M be a Kähler manifold, and $[\omega] \in H^2(M,\mathbb{R})$ its Kähler class. Suppose that ω is rational. **Then M is projective.**

COROLLARY: Let M be a non-projective compact Kähler manifold. **Then $H^{2,0}(M) \neq 0$,** in other words, M admits a holomorphic 2-form.

Proof: Assume that $H^{2,0}(M) = 0$. Then $H^{1,1}(M) = H^2(M)$. Since the Kähler cone is open in $H^{1,1}(M)$, it contains a rational point, hence M is projective. ∎

COROLLARY: Let M be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties. **Then its canonical class K_M is pseudoeffective.**

Proof: Consider a non-zero holomorphic 2-form $\Omega \in \Omega^2 M$, and let $\mathcal{F}$ be the kernel of Ω. Since the rank of Ω is 2, one has $\operatorname{rk} \mathcal{F} = 1$. **Brunella's theorem implies that F^* is pseudoeffective.** However, the normal bundle $TM/\mathcal{F}$ is equipped with a non-degenerate a 2-form Ω, hence its determinant vanishes. This gives $F^* = K_M$. ∎

Big and nef currents

DEFINITION: Let M be a compact, n-dimensional complex manifold. A positive, closed (1,1)-current θ is called **nef** if it is a limit of positive, closed $(1,1)$-forms. If the cohomology class $[\theta]$ in addition satisfies $\int_M [\theta]^n > 0$, then θ is called **big**.

REMARK: Let L be a holomorphic line bundle with $c_1(L)$ represented by a big and nef class (such a bundle is called **big and nef**). As follows from the Demailly's holomorphic Morse inequalities, the dimension $\dim H^0(L^N)$ grows as a polynomial of degree $\dim M$. Therefore, **a 3-manifold without non-trivial subvarieties has no big and nef bundles.**

THEOREM: Let θ be a positive (1,1)-current, with all Lelong sets of dimension 0. **Then θ is nef. If, in addition, its Lelong sets are non-empty, then θ is also big.**

Proof: Follows directly from Demailly's regularization theorem. ∎

Positive currents on 3-manifolds without subvarieties

REMARK: A positive current with 0-dimensional Lelong set **is cohomologous to a positive $(1,1)$-current which is smooth and strictly positive somewhere.** This follows from a simple argument based on **Demailly's regularized maximum** of plurisubharmonic functions.

COROLLARY: Let M be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties. **Then the canonical bundle K_M is nef, not big, and admits a singular metric with all Lelong numbers 0.**

Proof: As shown above, K_M admits a singular metric with pseudoeffective curvature θ. **All its Lelong sets have dimension 0, because M has no non-trivial subvarieties, hence θ is nef.** Since M has no divisors, K_M cannot be big, therefore **all Lelong numbers of θ vanish.** Indeed, a nef $(1,1)$-current which is strictly positive somewhere is also big, and a positive current with 0-dimensional Lelong set is cohomologous to a $(1,1)$-current which is strictly positive somewhere. ∎

Demailly-Peternell-Schneider vanishing theorem

The following theorem was rediscovered several times during 1990-ies (Enoki, Takegoshi, Morougane). Its most general form (which we use) is due to Demailly, Peternell and Schneider.

THEOREM: Let (M, I, ω) be a compact Kähler manifold, $\dim_{\mathbb{C}} M = n$, K its canonical bundle, and L a holomorphic line bundle on M equipped with a singular Hermitian metric h. Assume that the curvature Θ of L is a positive current on M, and denote by $\mathcal{I}(h)$ the corresponding multiplier ideal. Then **the wedge multiplication operator $\eta \longrightarrow \omega^i \wedge \eta$ induces a surjective map**

$$H^0(\Omega^{n-i} M \otimes L \otimes \mathcal{I}(h)) \xrightarrow{\omega^i \wedge \cdot} H^i(K \otimes L \otimes \mathcal{I}(h)).$$

Here ω is considered as an element in $H^1(\Omega^1 M)$, and multiplication by ω maps $H^k(\Omega^{n-l} M \otimes F)$ to $H^{k+1}(\Omega^{n-l+1} M \otimes F)$. ■

Demailly-Peternell-Schneider for threefolds without subvarieties

Demailly-Peternell-Schneider, applied to 3-manifolds without subvarieties, gives the following vanishing result.

COROLLARY: Let M be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties, and K its canonical bundle. **Then the multiplication map**

$$H^0(\Omega^{n-i}M \otimes K^m) \xrightarrow{\omega^i \wedge \cdot} H^i(K^{m+1})$$

is surjective for any $m \geqslant 0$.

Proof: The bundle K^m is pseudoeffective, and all its Lelong numbers vanish. Then the multiplier ideals for K^m are trivial. ■

Bounds on Euler characteristic

CLAIM: Let M be a complex manifold without divisors, and B a holomorphic bundle on M. **Then** $\dim H^0(B) \leqslant \operatorname{rk} B$**.**

Proof: Otherwise there would be a non-constant linear relation between sections; being non-constant, it must vanish on a divisor. ∎

PROPOSITION: Let M be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties, K its canonical bundle, and $A(m) := \chi(K^m)$ the holomorphic Euler characteristic of the pluricanonical bundle. **Then** $|A(m)| \leqslant 4$ **for any** $m \geqslant 1$**.**

Proof: From Demailly-Peternell-Schneider it follows that

$$-\dim H^0(\Omega^1 M \otimes K^{m-1}) - \dim H^0(\Omega^3 M \otimes K^{m-1}) \leqslant \\ \leqslant \chi(K^m) \leqslant \dim H^0(\Omega^0 M \otimes K^{m-1}) + \dim H^0(\Omega^2 M \otimes K^{m-1}).$$

The corresponding bundles have $\operatorname{rk} \leqslant 4$. By the above statement, their spaces of sections are no more than 4-dimensional. ∎

Riemann-Roch-Grothendieck and its applications

COROLLARY: Let M be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties, K_M its canonical bundle, and $A(m) := \chi(K_M^m)$ the holomorphic Euler characteristic of the pluricanonical bundle. **Then $A(m) = const$.**

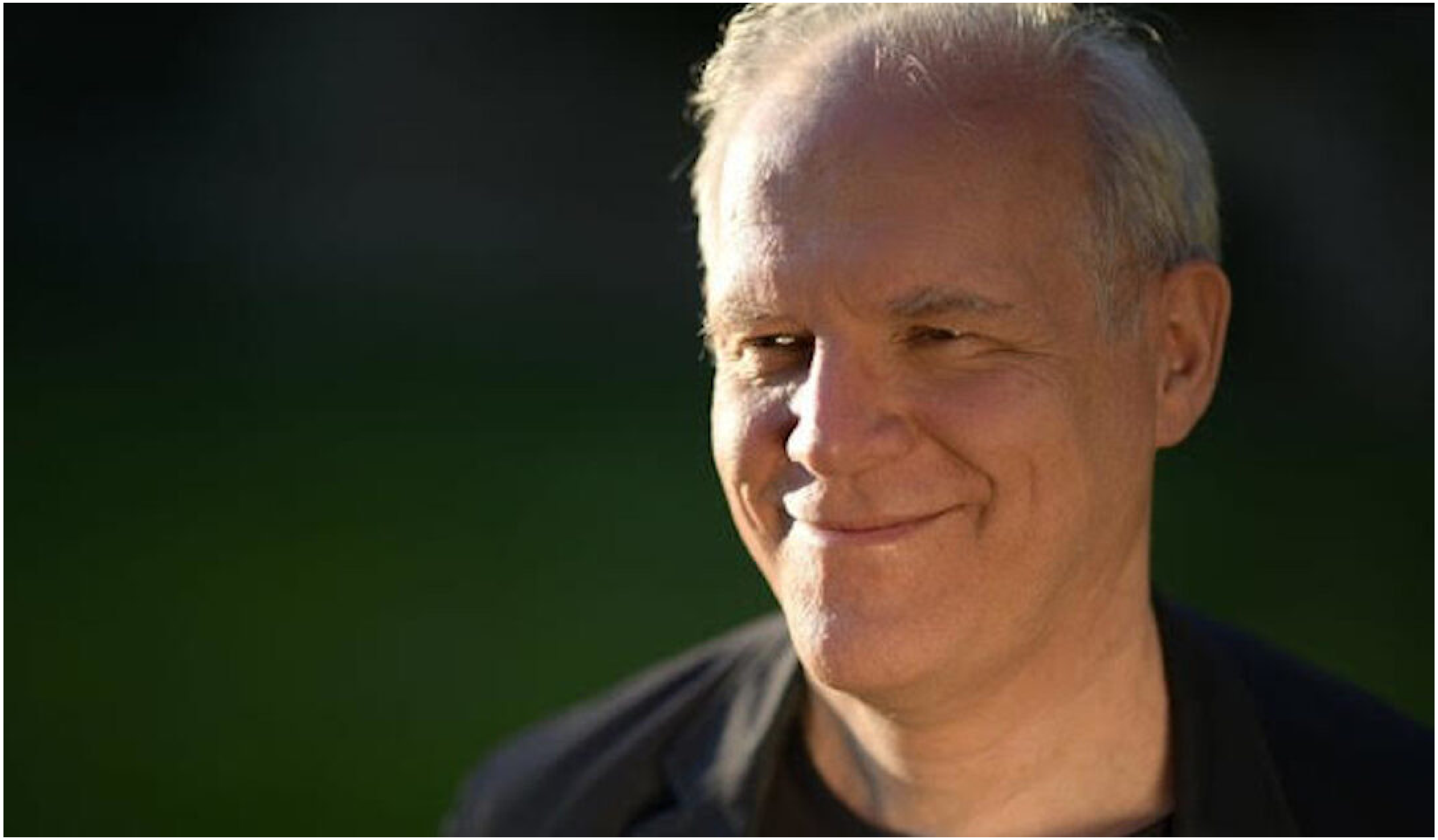
Proof: $A(m)$ is a polynomial by Riemann-Roch-Hirzebruch theorem, but it's bounded, hence constant. ■

The main result (Theorem 1) follows if we prove that M is Calabi-Yau.

COROLLARY: Let M be a 3-dimensional compact Kähler manifold which has no non-trivial compact complex subvarieties. **Then K_M is trivial.**

Proof. Step 1: $c_1(K_M)^3 = 0$, because K_M is nef and not big. Then Riemann-Roch-Hirzebruch formula gives $A(m) := \frac{(1-12m)}{24} c_1(M) c_2(M)$. Since $A(m)$ is bounded, this implies $c_1(M)c_2(M) = 0$, giving $A(m) = 0$.

Step 2: We obtained $\chi(O) = 1 - h^1(O) + h^2(O) - h^0(K_M) = 0$. Since M admits a holomorphic 2-form, $h^2(O) \geqslant 1$, which gives $h^0(K_M) \geqslant 1$. **However, a section of K_M cannot vanish anywhere, because M has no divisors.** This implies that K_M is trivial. ■



Jean-Pierre Demailly (1957-2022)