Global Torelli theorem for hyperkähler manifolds

Misha Verbitsky

Cycles, Calibrations and Nonlinear Partial Differential Equations

Celebrating Blaine Lawson's 70th Birthday

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Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

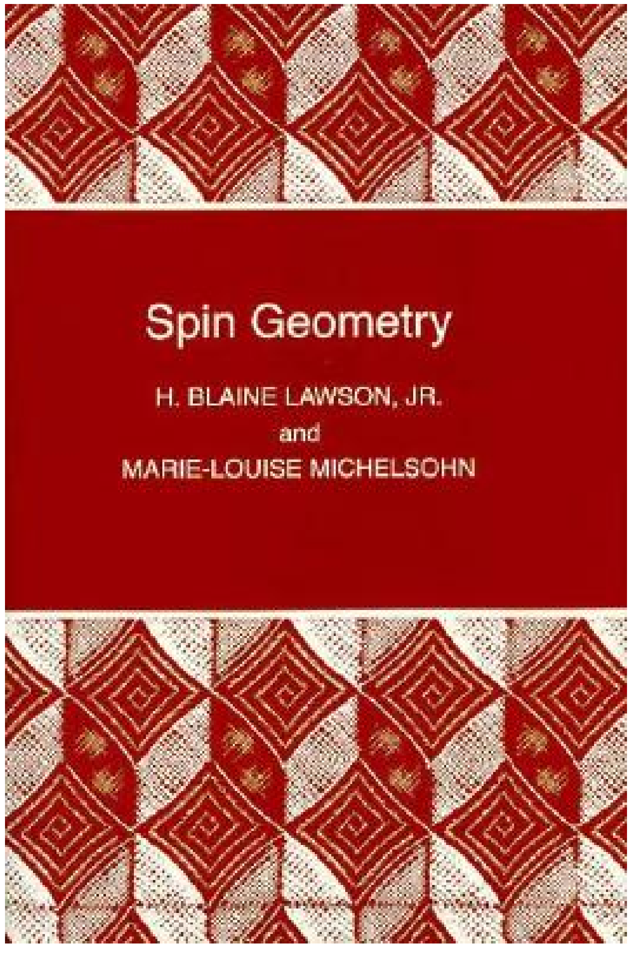
REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$**,** $\omega_J := g(J\cdot, \cdot)$**,** $\omega_K := g(K\cdot, \cdot)$**.**

DEFINITION: A **holomorphically symplectic manifold** is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. **Indeed,** $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ **is a holomorphic symplectic form on** (M, I)**.**

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric** in any Kähler class.

DEFINITION: For an algebraic geometer **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**


Spin Geometry
H. BLAINE LAWSON, JR.
and
MARIE-LOUISE MICHELSOHN

Torelli theorem for hyperkähler manifolds (plan):

1. Teichmüller space and the mapping class group

2. Algebraic geometries on calibrated manifolds

3. Topology of hyperkähler manifolds

4. Explicit computation of the mapping class group
(up to finite index subgroups)

5. Explicit computation of the Teichmüller space
(up to the non-Hausdorff points)

The Teichmüller space and the mapping class group

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\widetilde{\text{Teich}}$ the space of complex structures on M, and let $\text{Teich} := \widetilde{\text{Teich}}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **coarse moduli space of complex structures on M** is a connected component of Teich/Γ.

Remark: This terminology is **standard for curves.**

REMARK: To describe the moduli space, we shall compute Teich and Γ.

Remark: Even for a K3 surface, Teich/Γ is a **very bad space** (very much non-Hausdorff).

Calibrated geometries

by

REESE HARVEY and H. BLAINE LAWSON, JR.(1)

Rice University, Houston, TX, U.S.A.

University of California, Berkeley, CA, U.S.A.

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Calibrations

DEFINITION: Let $W \subset V$ be a p-dimensional subspace in a Euclidean space, and $\mathsf{Vol}(W)$ the Riemannian volume form of $W \subset V$. For any p-form $\eta \in \Lambda^p V$, **comass** $\mathsf{comass}(\eta)$ is the maximum of $\frac{\eta|_W}{\mathsf{Vol}(W)}$, for all p-dimensional subspaces $W \subset V$.

DEFINITION: A **calibration** on a Riemannian manifold M is a closed p-form with $\mathsf{comass}(\eta) \leqslant 1$ everywhere. A subspace $W \subset V$ is called **calibrated** if $\frac{\eta|_W}{\mathsf{Vol}(W)} = 1$.

DEFINITION: A calibrated subvariety is a p-dimensional subvariety $Z \subset M$ with singular subset of Hausdorff measure 0 and all tangent spaces $T_m Z$ at smooth points calibrated.

EXAMPLE: On a Kähler manifold (M, ω), **the form $\frac{\omega^p}{p!}$ is a calibration, and calibrated subvarieties are complex subvarieties.**

REMARK: One can define other complex-analytic notions in terms of a Kähler calibration, e.g. plurisubharmonic functions (Harvey-Lawson).

Classification of holonomies.

THEOREM: (de Rham) A complete, simply connected Riemannian manifold with non-irreducible holonomy **splits as a Riemannian product.**

THEOREM: (Berger, 1955) Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

ALL THE SPECIAL GEOMETRIES IN BERGER'S LIST ARE DEFINED BY CALIBRATIONS!

Calibrated algebraic geometry

SPECULATION: For many calibrated geometries, one can hypothesize the **"calibrated algebraic geometry"** associated with this particular calibration. The role of complex analysis is played by calibrated plurisubharmonic functions, complex subvarieties are replaced by the calibrated subvarieties, holomorphic bundles – by calibrated instantons etc.

REMARK: This program brings **hyperkähler algebraic geometry**.

THEOREM: (Grantcharov-V.) Let (M, I, J, K, g) be a hyperkähler manifold, $\omega_I, \omega_J, \omega_K$ the corresponding symplectic forms, and $\Theta_p := \frac{(\omega_I^2+\omega_J^2+\omega_K^2)^p}{c_p}$ the standard $SU(2)$-invariant $4p$-form normalized by $c_p = \sum_{k=1}^p \frac{(p!)^2}{(k!)^2}(2k)!4^{p-k}$. **Then Θ_p is a calibration, and its faces are p-dimensional quaternionic subspaces of TM.**

REMARK: The Θ_p-calibrated subvarities are **singular hyperkähler varieties**.

Trianalytic subvarieties

DEFINITION: Trianalytic subvarieties are closed subsets which are complex analytic with respect to I, J, K.

- Let L be a generic induced complex structure. Then **all complex subvarieties of (M, L) are trianalytic.**

- **A normalization of a trianalytic subvariety is smooth and hyperkähler.**

- A complex deformation of a trianalytic subvariety is again trianalytic, **the moduli space if its deformation is singular hyperkähler.**

Kähler cone

DEFINITION: A Kähler class of a Kähler manifold is the cohomology class of it Kähler form. The **Kähler cone** is the set of all Kähler classes.

MOTIVATION: To study the moduli spaces, we need to understand the Kähler cone of a hyperkähler manifold. **This is done using the Demailly-Paŭn theorem.**

THEOREM: (Demailly-Paŭn) Let M be a compact Kähler manifold, and $K \subset H^{1,1}(M,\mathbb{R})$ a subset consisting of all forms η such that $\int_Z \eta^p > 0$ for any p-dimensional complex subvariety $Z \subset M$. **Then the Kähler cone of M is one of the connected components of K.**

COROLLARY: Let M be a hyperkähler manifold with $H^{1,1}(M,\mathbb{Z}) = 0$. **Then its Kähler cone is a quadratic cone defined by a certain quadratic form.**

REMARK: The original work in that direction (done by Huybrechts) **used the duality argument, pioneered by Sullivan and Harvey-Lawson.**

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Inventiones mathematicae

An intrinsic characterization of Kähler manifolds

Reese Harvey* and H. Blaine Lawson, Jr.*

Department of Mathematics, Rice University, Post Office Box 1892, Houston, TX 77251, USA
Department of Mathematics, S.U.N.Y., Stony Brook, NY 11794, USA

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Simple hyperkähler manifolds

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

THEOREM: (V.) Let M be a simple hyperkähler manifold, b_2 its second Betti number. **Then the group** $\operatorname{Spin}(b_2-3, 3)$ **acts on the algebra** $H^*(M, \mathbb{R})$ **by automorphisms preserving the Pontryagin classes.** The corresponding action on $H^2(M, \mathbb{R})$ is the standard one.

COROLLARY: A 2-form on $H^2(M, \mathbb{R})$ is preserved!

THEOREM: (V.) The homomorphism $\operatorname{Aut}(H^*(M, \mathbb{R})) \longrightarrow O(H^2(M, \mathbb{R}))$ is surjective, and its kernel is a compact Lie group.

Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \\ - \frac{n-1}{n} \left(\int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n\right) \left(\int_X \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(b_2 - 3, 3)$.** It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

Computation of the mapping class group

Theorem: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

Theorem: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}),q)$ **has finite kernel.**

Proof: Follows from Sullivan and a computation of $\operatorname{Aut}(H^*(M,\mathbb{R}))$ done earlier. ■

DEFINITION: Two groups G, G' are called **commensurable** if G projects with finite kernel to a subgroup of finite index in G'.

DEFINITION: An **arithmetic group** is a group which is commensurable to an algebraic Lie group over integers.

COROLLARY: The mapping class group of a hyperkähler manifold **is an arithmetic group.**

The period map

Remark: For any $J \in \text{Teich}$, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let $P : \text{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map $P : \text{Teich} \longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

REMARK: P **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \ \mid \ q(l, l) = 0, q(l, \bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$

THEOREM: (Bogomolov) Let M be a simple hyperkähler manifold, and Teich its Teichmüller space. Then **The period map** $P : \text{Teich} \longrightarrow \mathbb{P}er$ **is etale.**

REMARK: Bogomolov's theorem implies that Teich **is smooth.** It is **non-Hausdorff** even in the simplest examples.

Hausdorff reduction

REMARK: A non-Hausdorff manifold is a topological space locally diffeomorphic to $\mathbb{R}^n$.

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (D. Huybrechts) If I_1, $I_2 \in$ Teich are non-separablee points, then $P(I_1) = P(I_2)$, and (M, I_1) **is birationally equivalent** to (M, I_2)

DEFINITION: Let M be a topological space for which $M/\sim$ is Hausdorff. Then $M/\sim$ is called **a Hausdorff reduction** of M.

Problems:

1. **$\sim$ is not always an equivalence relation.**
2. **Even if $\sim$ is equivalence, the $M/\sim$ is not always Hausdorff.**

Weakly Hausdorff manifolds

DEFINITION: A point $x \in X$ is called **Hausdorff** if $x \not\sim y$ for any $y \neq x$.

DEFINITION: Let M be an n-dimensional real analytic manifold, not necessarily Hausdoff. Suppose that **the set $Z \subset M$ of non-Hausdorff points is contained in a countable union of real analytic subvarieties** of codim $\geqslant 2$. Suppose, moreover, that

(S) For every $x \in M$, there is a closed neighbourhood $B \subset M$ of x and a continuous surjective map $\Psi : B \longrightarrow \mathbb{R}^n$ to a closed ball in $\mathbb{R}^n$, **inducing a homeomorphism** on an open neighbourhood of x.

Then M is called **a weakly Hausdorff manifold**.

REMARK: The period map satisfies (S). Also, the non-Hausdorff points of Teich **are contained in a countable union of divisors.**

THEOREM: A **weakly Hausdorff manifold X admits a Hausdorff reduction.** In other words, the quotient $X/\sim$ is a Hausdorff. Moreover, $X \longrightarrow X/\sim$ is locally a homeomorphism.

This theorem is proven using 1920-ies style point-set topology.

Birational Teichmüller moduli space

DEFINITION: The space $\mathrm{Teich}_b := \mathrm{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\mathrm{Teich}_b \xrightarrow{\mathrm{Per}} \mathbb{P}\mathrm{er}$ is an isomorphism, for each connected component of Teich_b.

DEFINITION: Let M be a hyperkaehler manifold, Teich_b its birational Teichmüller space, and Γ the mapping class group. The quotient Teich_b/Γ is called **the birational moduli space** of M.

REMARK: The birational moduli space is obtained from the usual moduli space **by gluing some (but not all) non-separable points. It is still non-Hausdorff.**

THEOREM: Let (M, I) be a hyperkähler manifold, and W a connected component of its birational moduli space. **Then W is isomorphic to $\mathbb{P}\mathrm{er}/\Gamma_I$, where $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ and Γ_I is an arithmetic group in $O(H^2(M, \mathbb{R}), q)$.**

A CAUTION: Usually "the global Torelli theorem" is understood as a theorem about Hodge structures. For K3 surfaces, **the Hodge structure on $H^2(M, \mathbb{Z})$ determines the complex structure**. For $\dim_{\mathbb{C}} M > 2$, **it is false**.

The birational Hodge-theoretic Torelli theorem

DEFINITION: The birational Hodge-theoretic Torelli theorem is true for M if Γ_I (the stabilizer of a Torelli component in the mapping class group) **is isomorphic to $O^+(H^2(M,\mathbb{Z}),q)$.**

REMARK: If a birational Hodge-theoretic Torelli theorem holds for M, then any deformation of M is up to a bimeromorphic equivalence **determined by the Hodge structure on $H^2(M)$.**

THEOREM: (Markman) The for $M = K3^{[n]}$, the group Γ_I **is a subgroup of $O^+(H^2(M,\mathbb{Z}),q)$ generated by oriented reflections.**

THEOREM: Let $M = K3^{[n+1]}$ with n a prime power. Then the (usual) global Torelli theorem holds birationally: **two deformations of a Hilbert scheme with isomorphic Hodge structures are bimeromorphic**. **For other n, it is false** (Markman).