Global Torelli theorem for hyperkähler manifolds

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Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M,I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

EXAMPLES.

EXAMPLE: An even-dimensional complex vector space.

EXAMPLE: An even-dimensional complex torus.

EXAMPLE: A non-compact example: $T^*\mathbb{C}P^n$ (Calabi).

REMARK: $T^*\mathbb{C}P^1$ **is a resolution of a singularity** $\mathbb{C}^2/\pm 1$.

EXAMPLE: Take a 2-dimensional complex torus T, then the singular locus of $T/\pm 1$ is of form $(\mathbb{C}^2/\pm 1) \times T$. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**.

REMARK: Take a symmetric square $\operatorname{Sym}^2 T$, with a natural action of T, and let $T^{[2]}$ be a blow-up of a singular divisor. **Then $T^{[2]}$ is naturally isomorphic to the Kummer surface $\widetilde{T/\pm 1}$.**

DEFINITION: A complex surface is called **K3 surface** if it a deformation of the Kummer surface.

THEOREM: (a special case of Enriques-Kodaira classification)
Let M be a compact complex surface which is hyperkähler. **Then M is either a torus or a K3 surface.**

Hilbert schemes

DEFINITION: A **Hilbert scheme** $M^{[n]}$ of a complex surface M is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension n over $\mathbb{C}$.

REMARK: A Hilbert scheme **is obtained as a resolution of singularities** of the symmetric power $\operatorname{Sym}^n M$.

THEOREM: (Fujiki, Beauville) **A Hilbert scheme of a hyperkähler surface is hyperkähler.**

EXAMPLE: A Hilbert scheme of K3 is hyperkähler.

EXAMPLE: Let T be a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, a universal covering of $T^{[n]}/T$ is called **a generalized Kummer variety**.

REMARK: There are 2 more "sporadic" examples of compact hyperkähler manifolds, constructed by K. O'Grady. **All known compact hyperkaehler manifolds are these 2 and the three series:** tori, Hilbert schemes of K3, and generalized Kummer.

The Teichmüller space and the mapping class group

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\widetilde{\operatorname{Teich}}$ the space of complex structures on M, and let $\operatorname{Teich} := \widetilde{\operatorname{Teich}}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira-Spencer-Kuranishi-Douady), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **coarse moduli space of complex structures on M** is a connected component of $\operatorname{Teich}/\Gamma$.

Remark: This terminology is **standard for curves.**

REMARK: For hyperkähler manifolds, it is convenient to take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler**. It is open in the usual Teichmüller space.

REMARK: To describe the moduli space, we shall compute Teich and Γ.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\begin{aligned}\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ -\frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)\end{aligned}$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: q **has signature** $(b_2-3,3)$**.** It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

Automorphisms of cohomology.

THEOREM: Let M be a simple hyperkähler manifold, and $G \subset GL(H^*(M))$ a group of automorphisms of its cohomology algebra preserving the Pontryagin classes. Then G acts on $H^2(M)$ **preserving the BBF form.** Moreover, the map $G \longrightarrow O(H^2(M,\mathbb{R}),q)$ **is surjective on a connected component, and has compact kernel.**

Proof. Step 1: Fujiki formula $v^{2n} = q(v,v)^n$ implies that Γ_0 **preserves the Bogomolov-Beauville-Fujiki up to a sign.** The sign is fixed, if n is odd.

Step 2: For even n, the sign is also fixed. Indeed, G preserves $p_1(M)$, and (as Fujiki has shown) $v^{2n-2} \wedge p_1(M) = q(v,v)^{n-1}c$, for some $c \in \mathbb{R}$. The constant c is positive, **because the degree of $c_2(B)$ is positive** for any Yang-Mills bundle with $c_1(B) = 0$.

Step 3: $\mathfrak{o}(H^2(M,\mathbb{R}),q)$ acts on $H^*(M,\mathbb{R})$ by derivations preserving Pontryagin classes (V., 1995). Therefore Lie(G) surjects to $\mathfrak{o}(H^2(M,\mathbb{R}),q)$.

Step 4: The kernel K of the map $G \longrightarrow G\big|_{H^2(M,\mathbb{R})}$ is compact, because it commutes with the Hodge decomposition and Lefschetz $\mathfrak{sl}(2)$-action, hence preserves the Riemann-Hodge form, which is positive definite. ∎

Computation of the mapping class group

Theorem: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

Theorem: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}),q)$ **has finite kernel.**

Proof: Follows from Sullivan and a computation of $\operatorname{Aut}(H^*(M,\mathbb{R}))$ done earlier. ■

REMARK: (Kollar-Matsusaka, Huybrechts) **There are only finitely many connected components** of Teich.

REMARK: The mapping class group acts on the set of connected components of Teich.

COROLLARY: Let Γ_I be the group of elements of mapping class group preserving a connected component of Teichmüller space containing $I \in$ Teich. **Then Γ_I is also arithmetic.** Indeed, **it has finite index in Γ.**

Deformations of holomorphically symplectic manifolds.

THEOREM: (Kodaira) **A small deformation of a compact Kähler manifold is again Kähler.**

COROLLARY: A small deformation of a holomorphically symplectic Kähler manifold M **is again holomorphically symplectic.**

Proof: A small deformation M' of M would satisfy $H^{2,0}(M') = H^{2,0}(M)$, however, a small deformation of a non-degenerate (2,0)-form remains non-degenerate. ■

COROLLARY: Small deformations of hyperkähler manifolds are hyperkähler.

REMARK: By **the moduli** of hyperkähler manifolds we shall understand the deformation space of complex manifolds admitting holomorphically symplectic and Kähler structure.

The period map

Remark: For any $J \in$ Teich, (M,J) is also a simple hyperkähler manifold, hence $H^{2,0}(M,J)$ is one-dimensional.

Definition: Let $P:$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map J to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map $P:$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called **the period map**.

REMARK: P maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \mid \ q(l,l) = 0, q(l,\bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$

THEOREM: Let M be a simple hyperkähler manifold, and Teich its Teichmüller space. Then
(i) (Bogomolov) **The period map $P:$ Teich $\longrightarrow \mathbb{P}er$ is etale.**
(ii) (Huybrechts) It is **surjective.**

REMARK: Bogomolov's theorem implies that Teich **is smooth.** It is **non-Hausdorff** even in the simplest examples.

Hausdorff reduction

REMARK: A non-Hausdorff manifold is a topological space locally diffeomorphic to $\mathbb{R}^n$.

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (D. Huybrechts) If I_1, $I_2 \in$ Teich are non-separablee points, then $P(I_1) = P(I_2)$, and (M, I_1) **is birationally equivalent** to (M, I_2)

DEFINITION: Let M be a topological space for which $M/\sim$ is Hausdorff. Then $M/\sim$ is called **a Hausdorff reduction** of M.

Problems:

1. **$\sim$ is not always an equivalence relation.**
2. **Even if $\sim$ is equivalence, the $M/\sim$ is not always Hausdorff.**

REMARK: A quotient $M/\sim$ is Hausdorff, if $M \longrightarrow M/\sim$ is open, and the graph $\Gamma_\sim \in M \times M$ is closed.

Weakly Hausdorff manifolds

DEFINITION: A point $x \in X$ is called **Hausdorff** if $x \not\sim y$ for any $y \neq x$.

DEFINITION: Let M be an n-dimensional real analytic manifold, not necessarily Hausdoff. Suppose that **the set $Z \subset M$ of non-Hausdorff points is contained in a countable union of real analytic subvarieties** of codim $\geqslant 2$. Suppose, moreover, that

(S) For every $x \in M$, there is a closed neighbourhood $B \subset M$ of x and a continuous surjective map $\Psi: \ B \longrightarrow \mathbb{R}^n$ to a closed ball in $\mathbb{R}^n$, **inducing a homeomorphism** on an open neighbourhood of x.

Then M is called **a weakly Hausdorff manifold**.

REMARK: The period map satisfies (S). Also, the non-Hausdorff points of Teich **are contained in a countable union of divisors.**

THEOREM: A **weakly Hausdorff manifold X admits a Hausdorff reduction.** In other words, the quotient $X/\sim$ is a Hausdorff. Moreover, $X \longrightarrow X/\sim$ is locally a homeomorphism.

This theorem is proven using 1920-ies style point-set topology.

Birational Teichmüller moduli space

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{\mathsf{Per}} \mathbb{P}\mathrm{er}$ is an isomorphism, for each connected component of Teich_b.

The proof is based on two results.

PROPOSITION: (The Covering Criterion) Let $X \xrightarrow{\varphi} Y$ be an etale map of smooth manifolds. Suppose that each $y \in Y$ has a neighbourhood $B \ni y$ diffeomorphic to a closed ball, such that for each connected component $B' \subset \varphi^{-1}(B)$, B' projects to B surjectively. **Then φ is a covering.**

PROPOSITION: The period map satisfies the conditions of the Covering Criterion.

Global Torelli theorem

DEFINITION: Let M be a hyperkaehler manifold, Teich_b its birational Teichmüller space, and Γ the mapping class group. The quotient $\operatorname{Teich}_b/\Gamma$ is called **the birational moduli space** of M.

REMARK: The birational moduli space is obtained from the usual moduli space **by gluing some (but not all) non-separable points. It is still non-Hausdorff.**

THEOREM: Let (M, I) be a hyperkähler manifold, and W a connected component of its birational moduli space. **Then W is isomorphic to $\mathbb{P}\mathrm{er}/\Gamma_I$, where $\mathbb{P}\mathrm{er} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ and Γ_I is an arithmetic group in $O(H^2(M, \mathbb{R}), q)$.**

A CAUTION: Usually "the global Torelli theorem" is understood as a theorem about Hodge structures. For K3 surfaces, **the Hodge structure on $H^2(M, \mathbb{Z})$ determines the complex structure**. For $\dim_{\mathbb{C}} M > 2$, **it is false**.

The birational Hodge-theoretic Torelli theorem

DEFINITION: The birational Hodge-theoretic Torelli theorem is true for M if Γ_I (the stabilizer of a Torelli component in the mapping class group) **is isomorphic to $O^+(H^2(M,\mathbb{Z}),q)$.**

REMARK: If a birational Hodge-theoretic Torelli theorem holds for M, then any deformation of M is up to a bimeromorphic equivalence **determined by the Hodge structure on $H^2(M)$.**

THEOREM: (Markman) The for $M = K3^{[n]}$, the group Γ_I **is a subgroup of $O^+(H^2(M,\mathbb{Z}),q)$ generated by oriented reflections.**

THEOREM: Let $M = K3^{[n+1]}$ with n a prime power. Then the (usual) global Torelli theorem holds birationally: **two deformations of a Hilbert scheme with isomorphic Hodge structures are bimeromorphic**. **For other n, it is false** (Markman).