Global Torelli theorem for hyperkähler manifolds

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Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $H^1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Remark: A simple hyperkähler manifold is always simply connected (Cheeger-Gromoll theorem).

Further on, all hyperkähler manifolds are assumed to be simple.

Deformations of holomorphically symplectic manifolds.

THEOREM: (Kodaira) **A small deformation of a compact Kähler manifold is again Kähler.**

COROLLARY: A small deformation of a holomorphically symplectic Kähler manifold M **is again holomorphically symplectic.**

Proof: A small deformation M' of M would satisfy $H^{2,0}(M') = H^{2,0}(M)$, however, a small deformation of a non-degenerate (2,0)-form remains non-degenerate. ■

COROLLARY: Small deformations of hyperkähler manifolds are hyperkähler.

REMARK: By **the moduli** of hyperkähler manifolds we shall understand the deformation space of complex manifolds admitting holomorphically symplectic and Kähler structure.

The Teichmüller space and the mapping class group

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\widetilde{\operatorname{Teich}}$ the space of complex structures on M, and let $\operatorname{Teich} := \widetilde{\operatorname{Teich}}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **coarse moduli space of complex structures on M** is a connected component of $\operatorname{Teich}/\Gamma$.

Remark: This terminology is **standard for curves.**

REMARK: For hyperkähler manifolds, it is convenient to take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler**. It is open in the usual Teichmüller space.

REMARK: To describe the moduli space, we shall compute Teich and Γ.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = \int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ - \frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(b_2 - 3, 3)$.** It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

Computation of the mapping class group

Theorem: (Sullivan) Let M be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map** $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ **has finite kernel, and its image has finite index in Γ_0.**

Theorem: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}),q)$ **has finite kernel.**

Proof. Step 1: Fujiki formula $v^{2n} = q(v,v)^n$ implies that Γ_0 **preserves the Bogomolov-Beauville-Fujiki up to a sign.** The sign is fixed, if n is odd.

Step 2: For even n, the sign is also fixed. Indeed, Γ_0 preserves $p_1(M)$, and (as Fujiki has shown) $v^{2n-2} \wedge p_1(M) = q(v,v)^{n-1}c$, for some $c \in \mathbb{R}$. The constant c is positive, **because the degree of $c_2(B)$ is positive** for any Yang-Mills bundle with $c_1(B) = 0$.

Computation of the mapping class group (cont.)

Step 3: $\mathfrak{o}(H^2(M,\mathbb{Q}),q)$ acts on $H^*(M,\mathbb{Q})$ by automorphisms preserving Pontryagin classes (V., 1995). Therefore $\Gamma_0\big|_{H^2(M,\mathbb{Q})}$ **is an arithmetic subgroup of $O(H^2(M,\mathbb{R}),q)$.**

Step 4: The kernel K of the map $\Gamma_0 \longrightarrow \Gamma_0\big|_{H^2(M,\mathbb{Q})}$ is finite, because it commutes with the Hodge decomposition and Lefschetz $\mathfrak{sl}(2)$-action, hence preserves the Riemann-Hodge form, which is positive definite. ■

REMARK: The same argument as in Step 4 also proves that **the group of automorphisms of $H^*(M,\mathbb{R})$ preserving p_1 is projected to $O(H^2(M,\mathbb{R}),q)$ with compact kernel**.

REMARK: (Kollar-Matsusaka, Huybrechts) **There are only finitely many connected components** of Teich.

REMARK: The mapping class group acts on the set of connected components of Teich.

COROLLARY: Let Γ_I be the group of elements of mapping class group preserving a connected component of Teichmüller space containing $I \in$ Teich. **Then Γ_I is also arithmetic.** Indeed, **it has finite index in Γ.**

The period map

Remark: For any $J \in$ Teich, (M, J) is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let P : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ map J to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map P : Teich $\longrightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called **the period map**.

REMARK: P maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$

THEOREM: Let M be a simple hyperkähler manifold, and Teich its Teichmüller space. Then
(i) (Bogomolov) **The period map P : Teich $\longrightarrow \mathbb{P}er$ is etale.**
(ii) (Huybrechts) It is **surjective.**

REMARK: Bogomolov's theorem implies that Teich **is smooth.** It is **non-Hausdorff** even in the simplest examples.

Hausdorff reduction

REMARK: **A non-Hausdorff manifold** is a topological space locally diffeomorphic to $\mathbb{R}^n$.

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (D. Huybrechts) If I_1, $I_2 \in$ Teich are non-separablee points, then $P(I_1) = P(I_2)$, and (M, I_1) **is birationally equivalent** to (M, I_2)

DEFINITION: Let M be a topological space for which $M/\sim$ is Hausdorff. Then $M/\sim$ is called **a Hausdorff reduction** of M.

Problems:

1. **$\sim$ is not always an equivalence relation.**
2. **Even if $\sim$ is equivalence, the $M/\sim$ is not always Hausdorff.**

REMARK: A quotient $M/\sim$ is Hausdorff, if $M \longrightarrow M/\sim$ is open, and the graph $\Gamma_\sim \in M \times M$ is closed.

Weakly Hausdorff manifolds

DEFINITION: A point $x \in X$ is called **Hausdorff** if $x \not\sim y$ for any $y \neq x$.

DEFINITION: Let M be an n-dimensional real analytic manifold, not necessarily Hausdoff. Suppose that **the set $Z \subset M$ of non-Hausdorff points is contained in a countable union of real analytic subvarieties** of codim $\geqslant 2$. Suppose, moreover, that

(S) For every $x \in M$, there is a closed neighbourhood $B \subset M$ of x and a continuous surjective map $\Psi: B \longrightarrow \mathbb{R}^n$ to a closed ball in $\mathbb{R}^n$, **inducing a homeomorphism** on an open neighbourhood of x.

Then M is called **a weakly Hausdorff manifold**.

REMARK: The period map satisfies (S). Also, the non-Hausdorff points of Teich **are contained in a countable union of divisors.**

THEOREM: A **weakly Hausdorff manifold X admits a Hausdorff reduction.** In other words, the quotient $X/\sim$ is a Hausdorff. Moreover, $X \longrightarrow X/\sim$ is locally a homeomorphism.

This theorem is proven using 1920-ies style point-set topology.

Kähler cone of a generic hyperkähler manifold

REMARK: Let M_1, M_2 be holomorphic symplectic manifolds, bimeromorphically equivalent. Then **$H^2(M_1)$ is naturally isomorphic to $H^2(M_2)$, and this isomorphism is compatible with Bogomolov-Beauville-Fujiki form.**

DEFINITION: A **modified nef cone** (also "birational nef cone" and "movable nef cone") is a closure of a union of all Kähler cones for all bimeromorphic models of a holomorphically symplectic manifold M.

THEOREM: (D. Huybrechts, S. Boucksom)
The modified nef cone is dual to the pseudoeffective cone under the Bogomolov-Beauville-Fujiki pairing.

Corollary: Let M be a simple hyperkähler manifold such that all integer (1,1)-classes satisfy $q(\nu,\nu) \geqslant 0$. **Then its Kähler cone is one of two components K_+ of a set $K := \{\nu \in H^{1,1}(M,\mathbb{R}) \mid q(\nu,\nu) > 0\}$.**

Proof: The pseudoeffective cone of M is contained in K_+ by divisorial Zariski decomposition. Therefore, the modified nef cone K_{MN} contains K_+. This means that $K_{MN} = K_+$. ■

Birational Teichmüller moduli space

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{\operatorname{Per}} \mathbb{P}\mathrm{er}$ is an isomorphism, for each connected component of Teich_b.

Sketch of a proof: 0. Since $\mathbb{P}\mathrm{er} = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$ **is simply connected, it will suffice to show that Per a covering.**

1. **It is etale** (Bogomolov).

2. For each hyperkähler structure (I,J,K) on M, **there is a whole S^2 of complex structures** $L = aI + bJ + cK$ on M, for $a^2+b^2+c^2=1$.

3. For any point $x \in \mathbb{P}\mathrm{er}$, let $NS(x)$ be the corresponding lattice of integer $(1,1)$-classes in $H^2(M)$. Then **the set $\mathcal{K} \subset H^{1,1}(M,\mathbb{R})$ of Kaehler classes can be determined explicitly** when $\operatorname{rk} NS(x) = 0$:

$$\mathcal{K} := \{\nu \in H^{1,1}(M,\mathbb{R}) \ \ | \ \ q(\nu,\nu) > 0\}$$

4. Every Kaehler class gives a hyperkaehler structure, hence a line in Teich_b. Such a line is called **generic hyperkähler curve** (GHK curve) if it passes through a point $x \in \mathbb{P}\mathrm{er}$ with $\operatorname{rk} NS(x) = 0$.

Birational Teichmüller moduli space (cont'd)

5. **For every such line C, $P^{-1}(C)$ is a disconnected union of rational curves bijectively mapped to C.** Indeed, for each $\tilde{x} \in$ Teich mapping to x, the set of GHK lines passing through $\tilde{x}$ is identified with $\mathcal{K}$, which is independent from the choice of $\tilde{x} \in \operatorname{Per}^{-1}(x)$.

6. **The whole Teichmüller space is covered by GHK curves.**

7. Surjectivity on GHK curves leads to the following condition (see the next slide).

(*) Let $\text{Teich}_b \xrightarrow{P} \mathbb{P}\text{er}$ be the period map. Then for each open subset $V \subset \mathbb{P}\text{er}$ with smooth boundary, and each connected component $W \subset P^{-1}(V)$, **the restriction $W \xrightarrow{P} V$ is surjective.**

8. The condition (*) **always implies that P is a covering.** It is, again, a (non-trivial) exercise in point-set topology.

9. The period space is simply connected, hence **P is an isomorphism on each connected component.**

Subtwistor metric on the Teichmüller space

DEFINITION: Let g be a Riemannian metric on Teich_b. Define **the subtwistor metric** d as the distance function $d(x,y)$ given by infimum of the length (in g) for all paths from x to y going through GHK curves.

CLAIM: The subtwistor metric induces the standard topology on any open subset $W \subset \operatorname{Teich}_b$.

Proof: The proof follows from Gleason-Palais-Montgomery classification of continuous groups.

THEOREM: Let $W \in \mathbb{P}\mathrm{er}$ be an open, connected subset, and $W_1 \subset \operatorname{Teich}_b$ a connected component of $\operatorname{Per}^{-1}(W)$. **Then $\operatorname{Per}: W_1 \longrightarrow W$ is surjective homeomorphism.**

Proof. Step 1: Whenever $x, y \in W$ are connected by segments of GHK curves which lie in W, and $x \in W_1$, one has $y \in W_1$, by lifting property of GHK curves. Therefore, **it would suffice to prove that all W is connected by segments of GHK curves** which lie in W.

Proof. Step 2: This is the same as to say that W is connected in the topology induced by subtwistor metric. By the previous claim, **this is equivalent to connectedness of W.** ■

Global Torelli theorem

DEFINITION: Let M be a hyperkaehler manifold, Teich_b its birational Teichmüller space, and Γ the mapping class group. The quotient $\operatorname{Teich}_b/\Gamma$ is called **the birational moduli space** of M.

REMARK: The birational moduli space is obtained from the usual moduli space **by gluing some (but not all) non-separable points. It is still non-Hausdorff.**

THEOREM: Let (M, I) be a hyperkähler manifold, and W a connected component of its birational moduli space. **Then W is isomorphic to $\mathbb{P}\mathrm{er}/\Gamma_I$, where $\mathbb{P}\mathrm{er} = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$ and Γ_I is an arithmetic group in $O(H^2(M,\mathbb{R}),q)$.**

A CAUTION: Usually "the global Torelli theorem" is understood as a theorem about Hodge structures. For K3 surfaces, **the Hodge structure on $H^2(M,\mathbb{Z})$ determines the complex structure**. For $\dim_{\mathbb{C}} M > 2$, **it is false**.

The marked moduli space

DEFINITION: Let Γ be the mapping class group, and $K \subset \Gamma$ the kernel of the natural map $\Gamma \longrightarrow GL(H^2(M,\mathbb{Z}))$. **It is finite,** as we have shown. The quotient Teich/K is called **the marked moduli space**.

THEOREM: The natural map $\operatorname{Teich} \longrightarrow \operatorname{Teich}/K$ **is a homeomorphism on each connected component.**

Proof. Step 1:
Let $I \in \operatorname{Teich}$ be a fixed point of a subgroup $K_I \subset K$. By Bogomolov's theorem, $T_I \operatorname{Teich}$ is naturally identified with $H^{1,1}_I(M)$. Since the action of K on $H^2(M)$ is trivial, any $\alpha \in K_I$ acts trivially on $T_I \operatorname{Teich}$. Therefore, K_I **acts as identity on a connected component of** Teich **containing** I.

Step 2: From Step 1, obtain that **the quotient map** $\operatorname{Teich} \xrightarrow{\Psi} \operatorname{Teich}/K$ **is a finite covering,** hence **it induces a finite covering of the corresponding Hausdorff reductions.** However, Ψ induces an isomorphism on each connected component of Teich_b, because **each component of** $\operatorname{Teich}_b = (\operatorname{Teich}_b)/K$ **is isomorphic to** $\mathbb{P}\mathrm{er}$. ■

The Hodge-theoretic Torelli theorem

REMARK: The group $O(p,q)$ ($p,q > 0$) has **4 connected components**, corresponding to the orientations of positive p-dimensional and negative q-dimensional planes.

DEFINITION: Let M be a hyperkaehler manifold. One says that **the Hodge-theoretic Torelli theorem holds for M** if

$$\operatorname{Teich}/\Gamma_I \longrightarrow \mathbb{P}\mathrm{er}/O^+(H^2(M,\mathbb{Z}),q),$$

where $O^+(H^2(M,\mathbb{Z}),q)$ is a subgroup of $O(H^2(M,\mathbb{Z}),q)$ preserving orientation on positive 3-planes. Equivalently, **it is true if M is uniquely determined by its Hodge structure**.

REMARK: The Hodge-theoretic Torelli theorem **is true for K3 surfaces**. **It is false** for all other known examples of hyperkaehler manifolds.

Problems:

1. The moduli space $\operatorname{Teich}/\Gamma$ **is not Hausdorff** (Debarre, 1984). Indeed, bimeromorphically equivalent hyperkähler manifolds have isomorphic Hodge structures.
2. **The covering $\operatorname{Teich}_b/\Gamma_I \longrightarrow \mathbb{P}\mathrm{er}/O^+(H^2(M,\mathbb{Z}),q)$ is non-trivial**, because the map $\Gamma_I \longrightarrow O^+(H^2(M,\mathbb{Z}),q)$ is not surjective (Namikawa, 2002).

The birational Hodge-theoretic Torelli theorem

DEFINITION: The birational Hodge-theoretic Torelli theorem is true for M if Γ_I (the stabilizer of a Torelli component in the mapping class group) **is isomorphic to $O^+(H^2(M,\mathbb{Z}),q)$.**

REMARK: If a birational Hodge-theoretic Torelli theorem holds for M, then any deformation of M is up to a bimeromorphic equivalence **determined by the Hodge structure on $H^2(M)$.**

THEOREM: (Markman) The for $M = K3^{[n]}$, the group Γ_I **is a subgroup of $O^+(H^2(M,\mathbb{Z}),q)$ generated by oriented reflections.**

THEOREM: Let $M = K3^{[n+1]}$ with n a prime power. Then the (usual) global Torelli theorem holds birationally: **two deformations of a Hilbert scheme with isomorphic Hodge structures are bimeromorphic**. **For other n, it is false** (Markman).