Global Torelli theorem for hyperkähler manifolds

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Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.**

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $H^1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Remark: A simple hyperkähler manifold is always simply connected (Cheeger-Gromoll theorem).

Further on, all hyperkähler manifolds are assumed to be simple.

Deformations of holomorphically symplectic manifolds.

REMARK: A Kaehler $2n$-manifold (M,ω) with $c_1(M) = 0$ and a holomorphic 2-form Ω which satisfies $\int_M \Omega^n \wedge \overline{\Omega}^n \neq 0$ **is holomorphically symplectic.** Indeed, if Ω is degenerate at Z, Ω^n is a section of canonical class vanishing at $Z \subset M$. Then $c_1(M) = [Z]$, but $\langle [Z], \omega^{n-1}\rangle = \int_Z \omega^{n-1} > 0$.

COROLLARY: A small deformation of a holomorphically symplectic manifold **is again holomorphically symplectic.**

THEOREM: (Kodaira) **A small deformation of a compact Kähler manifold is again Kähler.**

COROLLARY: **Small deformations of hyperkähler manifolds are hyperkähler.**

REMARK: By **the moduli** of hyperkähler manifolds we shall understand the deformation space of complex manifolds admitting holomorphically symplectic and Kähler structure.

The Teichmüller space and the mapping class group

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\widetilde{\operatorname{Teich}}$ the space of complex structures on M, and let $\operatorname{Teich} := \widetilde{\operatorname{Teich}}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **coarse moduli space of complex structures on M** is a connected component of $\operatorname{Teich}/\Gamma$.

Remark: This terminology is **standard for curves.**

REMARK: For hyperkähler manifolds, it is convenient to take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler**. It is open in the usual Teichmüller space.

REMARK: To describe the moduli space, we shall compute Teich and Γ.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some primitive integer quadratic form q on $H^2(M,\mathbb{Z})$, and $c > 0$ an integer number.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by the Fujiki's relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\begin{aligned}\lambda q(\eta,\eta) = &\int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ &-\frac{n-1}{n}\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)\end{aligned}$$

where Ω is the holomorphic symplectic form, and $\lambda > 0$.

Remark: **q has signature $(b_2 - 3, 3)$.** It is negative definite on primitive forms, and positive definite on $\langle \Omega, \overline{\Omega}, \omega\rangle$, where ω is a Kähler form.

Computation of the mapping class group

Theorem: (Sullivan) Let M be a compact simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ_0 the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in Γ_0.**

Theorem: Let M be a simple hyperkähler manifold, and Γ_0 as above. Then
(i) $\Gamma_0\big|_{H^2(M,\mathbb{Z})}$ **is a finite index subgroup of $O(H^2(M,\mathbb{Z}),q)$.**
(ii) The map $\Gamma_0 \longrightarrow O(H^2(M,\mathbb{Z}),q)$ **has finite kernel.**

Proof. Step 1: Fujiki formula $v^{2n} = q(v,v)^n$ implies that Γ_0 **preserves the Bogomolov-Beauville-Fujiki up to a sign.** The sign is fixed, if n is odd.

Step 2: For even n, the sign is also fixed. Indeed, Γ_0 preserves $p_1(M)$, and (as Fujiki has shown) $v^{2n-2} \wedge p_1(M) = q(v,v)^{n-1}c$, for some $c \in \mathbb{R}$. The constant c is positive, **because the degree of $c_2(B)$ is positive** for any Yang-Mills bundle with $c_1(B) = 0$.

Computation of the mapping class group (cont.)

Step 3: $\mathfrak{o}(H^2(M,\mathbb{Q}),q)$ acts on $H^*(M,\mathbb{Q})$ by automorphisms preserving Pontryagin classes (V., 1995). Therefore $\Gamma_0\big|_{H^2(M,\mathbb{Q})}$ **is an arithmetic subgroup of $O(H^2(M,\mathbb{R}),q)$.**

Step 4: The kernel K of the map $\Gamma_0 \longrightarrow \Gamma_0\big|_{H^2(M,\mathbb{Q})}$ is finite, because it commutes with the Hodge decomposition and Lefschetz $\mathfrak{sl}(2)$-action, hence preserves the Riemann-Hodge form, which is positive definite. ■

REMARK: The same argument as in Step 4 also proves that **the group of automorphisms of $H^*(M,\mathbb{R})$ preserving p_1 is projected to $O(H^2(M,\mathbb{R}),q)$ with compact kernel**.

REMARK: (Kollar-Matsusaka, Huybrechts) **There are only finitely many connected components** of Teich.

REMARK: The mapping class group acts on the set of connected components of Teich.

COROLLARY: Let Γ_I be the group of elements of mapping class group preserving a connected component of Teichmüller space containing $I \in$ Teich. **Then Γ_I is also arithmetic.** Indeed, **it has finite index in Γ**.

The period map

Remark: For any $J \in$ Teich, (M,J) is also a simple hyperkähler manifold, hence $H^{2,0}(M,J)$ is one-dimensional.

Definition: Let $P :$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map J to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map $P :$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called **the period map**.

REMARK: P maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \ | \ \ q(l,l) = 0, q(l,\bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2-3,3)/SO(2) \times SO(b_2-3,1)$

THEOREM: Let M be a simple hyperkähler manifold, and Teich its Teichmüller space. Then
(i) (Bogomolov) **The period map $P :$ Teich $\longrightarrow \mathbb{P}er$ is etale.**
(ii) (Huybrechts) It is **surjective.**

REMARK: Bogomolov's theorem implies that Teich **is smooth.** It is **non-Hausdorff** even in the simplest examples.

Hausdorff reduction

REMARK: A non-Hausdorff manifold is a topological space locally diffeomorphic to $\mathbb{R}^n$.

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

THEOREM: (D. Huybrechts) If I_1, $I_2 \in$ Teich are non-separablee points, then $P(I_1) = P(I_2)$, and (M, I_1) **is birationally equivalent** to (M, I_2)

DEFINITION: Let M be a topological space for which $M/\sim$ is Hausdorff. Then $M/\sim$ is called **a Hausdorff reduction** of M.

Problems:

1. **$\sim$ is not always an equivalence relation.**
2. **Even if $\sim$ is equivalence, the $M/\sim$ is not always Hausdorff.**

REMARK: A quotient $M/\sim$ is Hausdorff, if $M \longrightarrow M/\sim$ is open, and the graph $\Gamma_\sim \in M \times M$ is closed.

Weakly Hausdorff manifolds

DEFINITION: A point $x \in X$ is called **Hausdorff** if $x \not\sim y$ for any $y \neq x$.

DEFINITION: Let M be an n-dimensional real analytic manifold, not necessarily Hausdoff. Suppose that **the set $Z \subset M$ of non-Hausdorff points is contained in a countable union of real analytic subvarieties** of codim $\geqslant 2$. Suppose, moreover, that

(S) For every $x \in M$, there is a closed neighbourhood $B \subset M$ of x and a continuous surjective map $\Psi: B \longrightarrow \mathbb{R}^n$ to a closed ball in $\mathbb{R}^n$, **inducing a homeomorphism** on an open neighbourhood of x.

Then M is called **a weakly Hausdorff manifold**.

REMARK: The period map satisfies (S). Also, the non-Hausdorff points of Teich **are contained in a countable union of divisors.**

THEOREM: A **weakly Hausdorff manifold X admits a Hausdorff reduction.** In other words, the quotient $X/\sim$ is a Hausdorff. Moreover, $X \longrightarrow X/\sim$ is locally a homeomorphism.

Birational Teichmüller moduli space

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{P} \mathbb{P}er$ **is an isomorphism,** for each connected component of Teich_b.

Idea of a proof:

0. Since $\mathbb{P}er = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ **is simply connected, it will suffice to show it's a covering.**

1. **It is etale** (Bogomolov).

2. For each hyperkähler structure (I, J, K) on M, **there is a whole S^2 of complex structures** $L = aI + bJ + cK$ on M, for $a^2 + b^2 + c^2 = 1$.

Birational Teichmüller moduli space (cont'd)

3. For any point $x \in \mathbb{P}er$, let $NS(x)$ be the corresponding lattice of integer $(1,1)$-classes in $H^2(M)$. Then **the set of Kaehler classes can be determined explicitly** when $\operatorname{rk} NS(x) = 0$. Every Kaehler class gives a hyperkaehler structure, hence a line in Teich_b. Such a line is called **generic hyperkähler curve** (GHK curve).

4. **For every such line C, $P^{-1}(C)$ is a disconnected union of rational curves bijectively mapped to C.**

5. **The whole Teichmüller space is covered by GHK curves.**

6. Surjectivity on GHK curves leads to the following condition.

(*) Let $\text{Teich}_b \xrightarrow{P} \mathbb{P}er$ be the period map. Then for each open subset $V \subset \mathbb{P}er$ with smooth boundary, and each connected component $W \subset P^{-1}(V)$, **the restriction $W \xrightarrow{P} V$ is surjective.**

6. The condition (*) **always implies that P is a covering.**

7. The period space is simply connected, hence **P is an isomorphism on each connected component.**

Global Torelli theorem

DEFINITION: Let M be a hyperkaehler manifold, Teich_b its birational Teichmüller space, and Γ the mapping class group. The quotient $\operatorname{Teich}_b/\Gamma$ is called **the birational moduli space** of M.

REMARK: The birational moduli space is obtained from the usual moduli space **by gluing some (but not all) non-separable points. It is still non-Hausdorff.**

THEOREM: Let (M, I) be a hyperkähler manifold, and W a connected component of its birational moduli space. **Then W is isomorphic to $\mathbb{P}er/\Gamma_I$, where $\mathbb{P}er = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$ and Γ_I is an arithmetic group in $O(H^2(M,\mathbb{R}), q)$.**

A CAUTION: Usually "the global Torelli theorem" is understood as a theorem about Hodge structures. For K3 surfaces, **the Hodge structure on $H^2(M,\mathbb{Z})$ determines the complex structure**. For $\dim_{\mathbb{C}} M > 2$, **it is false**.

The Hodge-theoretic Torelli theorem

REMARK: The group $O(p,q)$ $(p,q > 0)$ has **4 connected components**, corresponding to the orientations of positive p-dimensional and negative q-dimensional planes.

DEFINITION: Let M be a hyperkaehler manifold. One says that **the Hodge-theoretic Torelli theorem holds for M** if

$$\mathrm{Teich}/\Gamma_I \longrightarrow \mathbb{P}er/O^+(H^2(M,\mathbb{Z}),q),$$

where $O^+(H^2(M,\mathbb{Z}),q)$ is a subgroup of $O(H^2(M,\mathbb{Z}),q)$ preserving orientation on positive 3-planes. Equivalently, **it is true if M is uniquely determined by its Hodge structure**.

REMARK: The Hodge-theoretic Torelli theorem **is true for K3 surfaces**. **It is false** for all other known examples of hyperkaehler manifolds.

Problems:
1. The moduli space Teich/Γ **is not Hausdorff** (Debarre, 1984). Indeed, bimeromorphically equivalent hyperkähler manifolds have isomorphic Hodge structures.
2. **The covering** $\mathrm{Teich}_b/\Gamma_I \longrightarrow \mathbb{P}er/O^+(H^2(M,\mathbb{Z}),q)$ **is non-trivial**, because the map $\Gamma_I \longrightarrow O^+(H^2(M,\mathbb{Z}),q)$ is not surjective (Namikawa, 2002).

The birational Hodge-theoretic Torelli theorem

DEFINITION: The birational Hodge-theoretic Torelli theorem is true for M if Γ_I (the stabilizer of a Torelli component in the mapping class group) **is isomorphic to $O^+(H^2(M,\mathbb{Z}),q)$.**

REMARK: If a birational Hodge-theoretic Torelli theorem holds for M, then any deformation of M is up to a bimeromorphic equivalence **determined by the Hodge structure on $H^2(M)$.**

THEOREM: (Markman) The for $M = K3^{[n]}$, the group Γ_I **is a subgroup of $O^+(H^2(M,\mathbb{Z}),q)$ generated by oriented reflections.**

THEOREM: Let $M = K3^{[n+1]}$ with n a prime power. Then the (usual) global Torelli theorem holds birationally: **two deformations of a Hilbert scheme with isomorphic Hodge structures are bimeromorphic**. **For other n, it is false** (Markman).