Global Torelli theorem for hyperkähler manifolds

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Hyperkähler manifolds

**DEFINITION:** A hyperkähler structure on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I,J,K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I,J,K$.

**REMARK:** A hyperkähler manifold has three symplectic forms $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

**REMARK:** This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves $I,J,K$.

**DEFINITION:** Let $M$ be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called the holonomy group of $M$.

**REMARK:** A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving $I,J,K$).
Global Torelli Theorem

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Holomorphically symplectic manifolds

**DEFINITION:** A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

**REMARK:** Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1} \omega_K$ is a holomorphic symplectic form on $(M, I)$.

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

**DEFINITION:** For the rest of this talk, a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.

**DEFINITION:** A hyperkähler manifold $M$ is called **simple** if $H^1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

**Bogomolov's decomposition:** Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

**Remark:** A simple hyperkähler manifold is always simply connected (Cheeger-Gromoll theorem).

Further on, all hyperkähler manifolds are assumed to be simple.
EXAMPLES.

EXAMPLE: An even-dimensional complex torus.

REMARK: Let $M$ be a 2-dimensional complex manifold with holomorphic symplectic form outside of singularities, which are all of form $\mathbb{C}^2/\pm 1$. Then its resolution is also holomorphically symplectic. It is isomorphic to $T^*\mathbb{C}P^1$.

EXAMPLE: Take a 2-dimensional complex torus $T$, then all the singularities of $T/\pm 1$ are of this form. Its resolution $\tilde{T}/\pm 1$ is called a Kummer surface. It is holomorphically symplectic.

REMARK: Take a symmetric square $\text{Sym}^2 T$, with a natural action of $T$, and let $T^{[2]}$ be a blow-up of a singular divisor. Then $T^{[2]}$ is naturally isomorphic to the Kummer surface $\tilde{T}/\pm 1$.

DEFINITION: A K3 surface is a complex 2-manifold obtained as a deformation of a Kummer surface.

REMARK: A K3 surface is always hyperkähler. Any hyperkähler manifold of real dimension 4 is isomorphic to a torus or a K3 surface.
Hilbert schemes

**DEFINITION:** A Hilbert scheme $M^{[n]}$ of a complex surface $M$ is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension $n$ over $\mathbb{C}$.

**REMARK:** A Hilbert scheme is obtained as a resolution of singularities of the symmetric power $\text{Sym}^n M$.

**THEOREM:** (Fujiki, Beauville) A Hilbert scheme of a hyperkähler surface is hyperkähler.

**EXAMPLE:** A Hilbert scheme of K3.

**EXAMPLE:** Let $T$ is a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, it is called a generalized Kummer variety.

**REMARK:** There are 2 more “sporadic” examples of compact hyperkähler manifolds, constructed by K. O’Grady. All known simple compact hyperkaehler manifolds are: Hilbert schemes of K3, generalized Kummer, and two O’Grady’s manifolds.
Deformations of holomorphically symplectic manifolds.

**REMARK:** A Kähler $2n$-manifold $(M, \omega)$ with $c_1(M) = 0$ and a holomorphic 2-form $\Omega$ which satisfies $\int_M \Omega^n \wedge \overline{\Omega}^n \neq 0$ is holomorphically symplectic. Indeed, if $\Omega$ is degenerate at $Z$, $\Omega^n$ is a section of canonical class vanishing at $Z \subset M$. Then $c_1(M) = [Z]$, but $\langle [Z], \omega^{n-1} \rangle = \int_Z \omega^{n-1} > 0$.

**COROLLARY:** A small deformation of a holomorphically symplectic manifold is again holomorphically symplectic.

**THEOREM:** (Kodaira) A small deformation of a compact Kähler manifold is again Kähler.

**COROLLARY:** Small deformations of hyperkähler manifolds are hyperkähler.

**REMARK:** By the moduli of hyperkähler manifolds we shall understand the deformation space of complex manifolds admitting holomorphically symplectic and Kähler structure.
**The Teichmüller space and the mapping class group**

**Definition:** Let $M$ be a compact complex manifold, and $\text{Diff}_0(M)$ a connected component of its diffeomorphism group (the group of isotopies). Denote by $\tilde{\text{Teich}}$ the space of complex structures on $M$, and let $\text{Teich} := \tilde{\text{Teich}}/\text{Diff}_0(M)$. We call it the **Teichmüller space**.

**Remark:** $\text{Teich}$ is a finite-dimensional complex space (Kodaira), but often non-Hausdorff.

**Definition:** Let $\text{Diff}_+(M)$ be the group of oriented diffeomorphisms of $M$. We call $\Gamma := \text{Diff}_+(M)/\text{Diff}_0(M)$ the **mapping class group**. The coarse moduli space of complex structures on $M$ is a connected component of $\text{Teich}/\Gamma$.

**Remark:** This terminology is standard for curves.

**REMARK:** For hyperkähler manifolds, it is convenient to take for $\text{Teich}$ the space of all complex structures of hyperkähler type, that is, holomorphically symplectic and Kähler. It is open in the usual Teichmüller space.

**REMARK:** To describe the moduli space, we shall compute $\text{Teich}$ and $\Gamma$. 
The Bogomolov-Beauville-Fujiki form

**THEOREM:** (Fujiki). Let \( \eta \in H^2(M) \), and \( \dim M = 2n \), where \( M \) is hyperkähler. Then \( \int_M \eta^{2n} = cq(\eta, \eta)^n \), for some primitive integer quadratic form \( q \) on \( H^2(M, \mathbb{Z}) \), and \( c > 0 \) an integer number.

**Definition:** This form is called **Bogomolov-Beauville-Fujiki form**. It is defined by the Fujiki’s relation uniquely, up to a sign. The sign is determined from the following formula (Bogomolov, Beauville)

\[
\lambda q(\eta, \eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \frac{n-1}{n} \left( \int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n} \right) \left( \int_X \eta \wedge \Omega^{n} \wedge \overline{\Omega}^{n-1} \right)
\]

where \( \Omega \) is the holomorphic symplectic form, and \( \lambda > 0 \).

**Remark:** \( q \) has signature \((b_2 - 3, 3)\). It is negative definite on primitive forms, and positive definite on \( \langle \Omega, \overline{\Omega}, \omega \rangle \), where \( \omega \) is a Kähler form.
Computation of the mapping class group

**Theorem:** (Sullivan) Let \( M \) be a compact simply connected Kähler manifold, \( \dim_{\mathbb{C}} M \geq 3 \). Denote by \( \Gamma_0 \) the group of automorphisms of an algebra \( H^*(M, \mathbb{Z}) \) preserving the Pontryagin classes \( p_i(M) \). Then the natural map \( \text{Diff}_+(M)/\text{Diff}_0 \longrightarrow \Gamma_0 \) has finite kernel, and its image has finite index in \( \Gamma_0 \).

**Theorem:** Let \( M \) be a simple hyperkähler manifold, and \( \Gamma_0 \) as above. Then

(i) \( \Gamma_0 \rvert_{H^2(M, \mathbb{Z})} \) is a finite index subgroup of \( O(H^2(M, \mathbb{Z}), q) \).

(ii) The map \( \Gamma_0 \longrightarrow O(H^2(M, \mathbb{Z}), q) \) has finite kernel.

**Proof. Step 1:** Fujiki formula \( v^{2n} = q(v, v)^n \) implies that \( \Gamma_0 \) preserves the Bogomolov-Beauville-Fujiki up to a sign. The sign is also fixed, if \( n \) is odd.

**Step 2:** For even \( n \), the sign is also fixed. Indeed \( \Gamma_0 \) preserves \( p_1(M) \), and (as Fujiki has shown) \( v^{2n-2} \wedge p_1(M) = q(v, v)^{n-1}c \), for some \( c \in \mathbb{R} \). The constant \( c \) is positive, because the degree of \( c_2(B) \) is positive for any Yang-Mills bundle with \( c_1(B) = 0 \).
Computation of the mapping class group (cont.)

Step 3: $\sigma(H^2(M, \mathbb{Q}), q)$ acts on $H^*(M, \mathbb{Q})$ by automorphisms preserving Pontryagin classes (V., 1995). Therefore $\Gamma_0|_{H^2(M, \mathbb{Q})}$ is an arithmetic subgroup of $O(H^2(M, \mathbb{R}), q)$.

Step 4: The kernel $K$ of the map $\Gamma_0 \rightarrow \Gamma_0|_{H^2(M, \mathbb{Q})}$ is finite, because it commutes with the Hodge decomposition and Lefschetz $sl(2)$-action, hence preserves the Riemann-Hodge form, which is positive definite. ■

REMARK: The same argument as in Step 4 also proves that the group of automorphisms of $H^*(M, \mathbb{R})$ preserving $p_1$ is projected to $O(H^2(M, \mathbb{R}), q)$ with compact kernel.

REMARK: (Kollar-Matsusaka, Huybrechts) There are only finitely many connected components of Teich.

REMARK: The mapping class group acts on the set of connected components of Teich.

COROLLARY: Let $\Gamma_I$ be the group of elements of mapping class group preserving a connected component of Teichmüller space containing $I \in \text{Teich}$. Then $\Gamma_I$ is also arithmetic. Indeed, it has finite index in $\Gamma$. 

The period map

**Remark:** For any \( J \in \text{Teich} \), \((M, J)\) is also a simple hyperkähler manifold, hence \( H^{2,0}(M, J) \) is one-dimensional.

**Definition:** Let \( P : \text{Teich} \rightarrow \mathbb{P}H^2(M, \mathbb{C}) \) map \( J \) to a line \( H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C}) \). The map \( P : \text{Teich} \rightarrow \mathbb{P}H^2(M, \mathbb{C}) \) is called the period map.

**REMARK:** \( P \) maps Teich into an open subset of a quadric, defined by
\[
\text{Per} := \{ l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0 \}.
\]

It is called the period space of \( M \).

**REMARK:** \( \text{Per} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1) \)

**THEOREM:** Let \( M \) be a simple hyperkähler manifold, and Teich its Teichmüller space. Then

(i) (Bogomolov) **The period map** \( P : \text{Teich} \rightarrow \text{Per} \) is etale.

(ii) (Huybrechts) It is surjective.

**REMARK:** Bogomolov’s theorem implies that Teich is smooth. It is non-Hausdorff even in the simplest examples.
Hausdorff reduction

**REMARK:** A non-Hausdorff manifold is a topological space locally diffeomorphic to $\mathbb{R}^n$.

**DEFINITION:** Let $M$ be a topological space. We say that $x, y \in M$ are non-separable (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

**THEOREM:** (D. Huybrechts) If $I_1, I_2 \in \text{Teich}$ are non-separable points, then $P(I_1) = P(I_2)$, and $(M, I_1)$ is birationally equivalent to $(M, I_2)$

**DEFINITION:** Let $M$ be a topological space for which $M/\sim$ is Hausdorff. Then $M/\sim$ is called a **Hausdorff reduction** of $M$.

**Problems:**
1. $\sim$ is not an equivalence relation.
2. Even if $\sim$ is equivalence, the $M/\sim$ is not always Hausdorff.

**REMARK:** A quotient $M/\sim$ is Hausdorff, if $M \rightarrow M/\sim$ is open, and the graph $\Gamma_{\sim} \in M \times M$ is closed.
Weakly Hausdorff manifolds

**DEFINITION:** A point \( x \in X \) is called **Hausdorff** if \( x \not\sim y \) for any \( y \neq x \).

**DEFINITION:** Let \( M \) be an \( n \)-dimensional real analytic manifold, not necessarily Hausdorff. Suppose that the set \( Z \subset M \) of non-Hausdorff points is contained in a countable union of real analytic subvarieties of codim \( \geq 2 \). Suppose, moreover, that

(S) For every \( x \in M \), there is a closed neighbourhood \( B \subset M \) of \( x \) and a continuous surjective map \( \Psi : B \to \mathbb{R}^n \) to a closed ball in \( \mathbb{R}^n \), inducing a **homeomorphism** on an open neighbourhood of \( x \).

Then \( M \) is called a **weakly Hausdorff manifold**.

**REMARK:** The period map satisfies \((S)\). Also, the non-Hausdorff points of Teich are contained in a countable union of divisors.

**THEOREM:** A weakly Hausdorff manifold \( X \) admits a Hausdorff reduction. In other words, the quotient \( X/\sim \) is a Hausdorff. Moreover, \( X \to X/\sim \) is locally a homeomorphism.
Birational Teichmüller moduli space

**DEFINITION:** The space \( \text{Teich}_b := \text{Teich} / \sim \) is called the birational Teichmüller space of \( M \).

**THEOREM:** The period map \( \text{Teich}_b \xrightarrow{P} \text{Per} \) is an isomorphism, for each connected component of \( \text{Teich}_b \).

**Idea of a proof:**

0. Since \( \text{Per} = SO(b_2 - 3, 3) / SO(2) \times SO(b_2 - 3, 1) \) is simply connected, it will suffice to show it's a covering.

1. **It is étale** (Bogomolov).

2. For each hyperkähler structure \( (I, J, K) \) on \( M \), there is a whole \( S^2 \) of complex structures \( L = aI + bJ + cK \) on \( M \), for \( a^2 + b^2 + c^2 = 1 \).

3. On every such line, \( P \) is surjective.
Birational Teichmüller moduli space (cont’d)

4. For any point \( x \in \text{Per} \), let \( NS(x) \) be the corresponding lattice of integer \((1,1)\)-classes in \( H^2(M) \). Then the set of Kaehler classes can be determined explicitly when \( \text{rk} \ NS(x) = 0 \). Every Kaehler class gives a hyperkaehler structure, hence a line in \( \text{Teich}_b \). Such a line is called generic hyperkähler curve (GHK curve).

5. The whole Teichmüller space is covered by GHK curves.

6. Surjectivity on GHK curves leads to the following condition.

(*) Let \( \text{Teich}_b \xrightarrow{P} \text{Per} \) be the period map. Then for each open subset \( V \subset \text{Per} \) with smooth boundary, and each connected component \( W \subset P^{-1}(V) \), the restriction \( W \xrightarrow{P} V \) is surjective.

6. The condition (*) always implies that \( P \) is a covering.

7. The period space is simply connected, hence \( P \) is an isomorphism on each connected component.
Global Torelli theorem

**DEFINITION:** Let $M$ be a hyperkaehler manifold, $\text{Teich}_b$ its birational Teichmüller space, and $\Gamma$ the mapping class group. The quotient $\text{Teich}_b/\Gamma$ is called the **birational moduli space** of $M$.

**REMARK:** The birational moduli space is obtained from the usual moduli space by gluing some (but not all) non-separable points. It is still non-Hausdorff.

**THEOREM:** Let $(M, I)$ be a hyperkähler manifold, and $W$ a connected component of its birational moduli space. Then $W$ is isomorphic to $\mathbb{P}er/\Gamma_I$, where $\mathbb{P}er = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ and $\Gamma_I$ is an arithmetic group in $O(H^2(M, \mathbb{R}), q)$.

**A CAUTION:** Usually “the global Torelli theorem” is understood as a theorem about Hodge structures. For K3 surfaces, **the Hodge structure on $H^2(M, \mathbb{Z})$ determines the complex structure**. For $\dim_{\mathbb{C}} M > 2$, it is false.
The Hodge-theoretic Torelli theorem

**REMARK:** The group $O(p,q)$ ($p,q > 0$) has **4 connected components**, corresponding to the orientations of positive $p$-dimensional and negative $q$-dimensional planes.

**DEFINITION:** Let $M$ be a hyperkähler manifold. One says that the **Hodge-theoretic Torelli theorem holds for** $M$ if

$$\text{Teich} / \Gamma_I \longrightarrow \mathbb{P}er / O^+(H^2(M,\mathbb{Z}),q),$$

where $O^+(H^2(M,\mathbb{Z}),q)$ is a subgroup of $O(H^2(M,\mathbb{Z}),q)$ preserving orientation on positive 3-planes. Equivalently, **it is true if** $M$ **is uniquely determined by its Hodge structure.**

**REMARK:** The Hodge-theoretic Torelli theorem **is true for K3 surfaces.** **It is false** for all other known examples of hyperkähler manifolds.

**Problems:**
1. The moduli space $\text{Teich} / \Gamma$ **is not Hausdorff** (Debarre, 1984). Indeed, bimeromorphically equivalent hyperkähler manifolds have isomorphic Hodge structures.
2. **The covering** $\text{Teich}_b / \Gamma_I \longrightarrow \mathbb{P}er / O^+(H^2(M,\mathbb{Z}),q)$ **is non-trivial**, because the map $\Gamma_I \longrightarrow O^+(H^2(M,\mathbb{Z}),q)$ is not surjective (Namikawa, 2002).
The birational Hodge-theoretic Torelli theorem

**DEFINITION:** The birational Hodge-theoretic Torelli theorem is true for $M$ if $\Gamma_I$ (the stabilizer of a Torelli component in the mapping class group) is isomorphic to $O^+(H^2(M,\mathbb{Z}),q)$.

**REMARK:** If a birational Hodge-theoretic Torelli theorem holds for $M$, then any deformation of $M$ is up to a bimeromorphic equivalence determined by the Hodge structure on $H^2(M)$.

**THEOREM:** (Markman) The for $M = \text{K3}^n$, the group $\Gamma_I$ is a subgroup of $O^+(H^2(M,\mathbb{Z}),q)$ generated by oriented reflections.

**THEOREM:** Let $M = \text{K3}^{n+1}$ with $n$ a prime power. Then the (usual) global Torelli theorem holds birationally: two deformations of a Hilbert scheme with isomorphic Hodge structures are bimeromorphic. For other $n$, it is false (Markman).