Global Torelli theorem for hyperkaehler manifolds

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Hyperkähler manifolds

DEFINITION: A **hyperkaehler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kaehler for I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

REMARK: A hyperkähler manifold holomorphically symplectic: $\Omega := \omega_J + \sqrt{-1}\,\omega_K$ is **non-degenerate, holomorphic, closed (2,0)-form** on (M, I).

Calabi-Yau theorem gives a unique Ricci-flat Kähler metric on M, in any Kähler class, if $c_1(M) = 0$. If M is also holomorphically symplectic, this metric is **hyperkähler**. It follows from Bochner's vanishing, Berger's classification of irreducible holonomy groups, and de Rham's decomposition theorem.

THEOREM: A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkaehler metric in every Kaehler class.**

REMARK: **In the sequel, "hyperkaehler manifold" is understood as "compact, Kähler, holomorphically symplectic".**

Simple Hyperkähler manifolds

Definition: A hyperkähler manifold M is called **simple** if $H^1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

REMARK: A hyperkaehler manifold is one with (global) holonomy in $Sp(n)$. **It is simple if its local holonomy is $Sp(n)$.**

THEOREM: (Bogomolov's decomposition): Any hyperkähler manifold admits a finite covering, which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. Then $\int_M \eta^{2n} = q(\eta,\eta)^n$, for some integer quadratic form q on $H^2(M)$.

Definition: This form is called **Bogomolov-Beauville-Fujiki form**. **It is defined by this relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$cq(\eta,\eta) = (n/2)\int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ -(1-n)\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $c = C^{n-1}_{2n-2}\lambda \int_M \wedge\Omega^n\wedge\overline{\Omega}^n > 0$
.

Remark: q **has signature** $(b_2-3,3)$**.** It is negative definite on primitive forms, and positive definite on $\langle\Omega,\overline{\Omega},\omega\rangle$, where ω is a Kähler form.

The Teichmuller space and the mapping class group

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\widetilde{\operatorname{Teich}}$ the space of complex structures on M, and let $\operatorname{Teich} := \widetilde{\operatorname{Teich}}/\operatorname{Diff}_0(M)$. We call it **the Teichmuller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **coarse moduli space of complex structures on M** is a connected component of $\operatorname{Teich}/\Gamma$.

Remark: This terminology is **standard for curves.**

REMARK: For hyperkaehler manifolds, it is convenient to take for Teich **the space of all complex structures of hyperkaehler type**, that is, holomorphically symplectic and Kaehler. It is open in the usual Teichmüller space.

The period map

Remark: For any $J \in$ Teich, (M,J) is also a simple hyperkähler manifold, hence $H^{2,0}(M,J)$ is one-dimensional.

Definition: Let P : Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map J to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map $P:$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called **the period map**.

Remark: P maps Teich into an open subset of a quadric, defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \mid \ q(l,l) = 0, q(l,\bar{l}) > 0.$$

It is called **the period space** of M.

THEOREM: Let M be a simple hyperkaehler manifold, and Teich its Teichmuller space. Then
(i) (Bogomolov) **The period map $P:$ Teich $\longrightarrow \mathbb{P}er$ is etale.**
(ii) (Huybrechts) It is **surjective.**

Remark: Bogomolov's theorem implies that Teich **is smooth.**

THEOREM: (D. Huybrechts) If I_1, $I_2 \in$ Teich are non-separate points, then $P(I_1) = P(I_2)$, and (M,I_1) **is birationally equivalent** to (M,I_2)

Hausdorff reduction

REMARK: A non-Hausdorff manifold is a topological space locally diffeomorphic to $\mathbb{R}^n$.

DEFINITION: Let X be a topological space, and $X \xrightarrow{\varphi} X_0$ a continuous surjection. The space X_0 is called **a Hausdorff reduction** of X if any continuous map $X \longrightarrow X'$ to a Hausdorff space is factorized through φ.

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

CLAIM: A Hausdorff reduction, when it exists, is isomorphic to a quotient $M/\sim$.

Problems:

1. **$\sim$ is not an equivalence relation.**
2. **Even if $\sim$ is equivalence, the $M/\sim$ is not always Hausdorff.**

REMARK: A quotient $M/\sim$ is Hausdorff, if $M \longrightarrow M/\sim$ is open, and the graph $\Gamma_\sim \in M \times M$ is closed.

Global Torelli theorem

THEOREM: Let P : Teich $\longrightarrow \mathbb{P}er$ be a period map of a hyperkaehler manifold. **Then P is a Hausdorff reduction,** for each connected component of Teich. In particular, it is surjective and generically one-to-one on each component.

A CAUTION: Usually "the global Torelli theorem" is understood as a theorem about Hodge structures. For K3 surfaces, **the Hodge structure on $H^2(M,\mathbb{Z})$ determines the complex structure** (Tyurina, Shafarevich, Pyatetski-Shapiro, Burns, Rappoport, Todorov, Siu, ...). **This can be understood as isomorphism**

$$\text{Teich}\,/\Gamma_I \longrightarrow \mathbb{P}er/O^+(H^2(M,\mathbb{Z}),q).$$

Here Γ_I is a subgroup of the mapping class group stabilizing a connected component. For $\dim_{\mathbb{C}} M > 2$, **it is false**.

Problems:

1. The moduli space Teich$\,/\Gamma$ **is not Hausdorff** (Debarre, 1984).
2. **The covering** Teich $/\Gamma_I \longrightarrow \mathbb{P}er/O^+(H^2(M,\mathbb{Z}),q)$ **is non-trivial**, because the map $\Gamma_I \longrightarrow O^+(H^2(M,\mathbb{Z}),q)$ is not surjective (Namikawa, 2002).

The Hodge-theoretic Torelli theorem

REMARK: The group $O(p,q)$ $(p,q>0)$ has **4 connected components**, corresponding to the orientations of positive p-dimensional and negative q-dimensional planes.

DEFINITION: A spin-orientation on a space with signature p,q is a choice of orientation of positive p-dimensional planes. The group preserving spin-orientation is denoted by $O^+(p,q)$.

REMARK: For the $H^2(M)$, **the spin orientation is determined by the Kaehler and holomorphic symplectic forms.**

REMARK: A birational Hodge-theoretic Torelli theorem is true if and only if Γ_I (the stabilizer of a Torelli component in the mapping class group) **is isomorphic to $O^+(H^2(M,\mathbb{Z}),q)$.**

THEOREM: (Markman) The for $M=K3^{[n+1]}$, the group Γ_I **is a subgroup of $O^+(H^2(M,\mathbb{Z}),q)$ generated by spin-oriented reflections.**

THEOREM: Let $M=K3^{[n+1]}$ with n a prime power. Then the (usual) global Torelli theorem holds birationally: **two deformations of a Hilbert scheme with isomorphic Hodge structures are bimeromorphic**. **For other n, it is false** (Markman).

Computation of the mapping class group

Theorem: (Sullivan) Let M be a compact simply connected (or nilpotent) Kaehler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma$ has finite kernel, and its image has finite index in Γ.**

Theorem: Let M be a simple hyperkaehler manifold, and Γ as above. Then
(i) $\Gamma\big|_{H^2(M,\mathbb{Q})}$ **is an arithmetic subgroup** of $O(H^2(M,\mathbb{Q}),q)$.
(ii) The map $\Gamma \longrightarrow O(H^2(M,\mathbb{Q}),q)$ **has finite kernel.**

Proof. Step 1: Fujiki formula $v^{2n} = q(v,v)^n$ implies that Γ **preserves the Bogomolov-Beauville-Fujiki up to a sign.**

Step 2: The sign is fixed, because Γ preserves $p_1(M)$, and (as Fujiki has shown) $v^{2n-2} \wedge p_1(M) = q(v,v)^{n-1}c$, for some $c \in \mathbb{R}$. The constant c is positive, **because the degree of $c_2(B)$ is positive** for any Yang-Mills bundle with $c_1(B) = 0$.

Step 3: $\mathfrak{o}(H^2(M,\mathbb{Q}),q)$ acts on $H^*(M,\mathbb{Q})$ by automorphisms preserving Pontryagin classes (V., 1995). Therefore $\Gamma\big|_{H^2(M,\mathbb{Q})}$ **is an arithmetic subgroup of $O(H^2(M,\mathbb{R}),q)$.**

Computation of the mapping class group (cont.)

Step 4: The kernel K of the map $\Gamma \longrightarrow \Gamma\Big|_{H^2(M,\mathbb{Q})}$ is finite, because it commutes with the Hodge decomposition and Lefschetz $\mathfrak{sl}(2)$-action, hence preserves the Riemann-Hodge form, which is positive definite. ■

REMARK: The same argument also proves that **the group of automorphisms of $H^*(M,\mathbb{R})$ preserving p_1 is projected to $O(H^2(M,\mathbb{R}),q)$ with compact kernel**.

REMARK: (Kollar-Matsusaka, Huybrechts) **There are only finitely many connected components** of Teich.

COROLLARY: The group Γ_I of elements of mapping class group preserving a Torelli component is also arithmetic.

THEOREM: Let M be a hyperkaehler manifold, and W its birational moduli space. **Then W is isomorphic to $\mathbb{P}er/\Gamma_I$, where $\mathbb{P}er = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$ and Γ_I is an arithmetic group in $O(H^2(M,\mathbb{R}),q)$**

Weakly Hausdorff manifolds

DEFINITION: A point $x \in X$ is called **Hausdorff** if $x \not\sim y$ for any $y \neq x$.

DEFINITION: Let M be an n-dimensional real analytic manifold, not necessarily Hausdoff. Suppose that **the set $Z \subset M$ of non-Hausdorff points is contained in a countable union of real analytic subvarieties** of codim $\geqslant 2$. Suppose, moreover, that

(S) For every $x \in M$, there is a closed neighbourhood $B \subset M$ of x and a continuous surjective map $\Psi: B \longrightarrow \mathbb{R}^n$ to a closed ball in $\mathbb{R}^n$, **inducing a homeomorphism** on an open neighbourhood of x.

Then M is called **a weakly Hausdorff manifold**.

REMARK: The period map satisfies (S). Also, the non-Hausdorff points of Teich **are contained in the countable union of divisors.**

THEOREM: A **weakly Hausdorff manifold X admits a Hausdorff reduction.** In other words, the quotient $X/\sim$ is a Hausdorff. Moreover, $X \longrightarrow X/\sim$ is locally a homeomorphism.

Birational Teichmuller moduli space

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmuller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{P} \mathbb{P}er$ **is an isomorphism.**

Idea of a proof:

0. Since $\mathbb{P}er = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ **is simply connected, it will suffice to show it's a covering.**

1. **It is etale** (Bogomolov).

2. For each hyperkaehler structure (I, J, K) on M, **there is a whole** S^2 **of complex structures** $L = aI + bJ + cK$ on M, for $a^2 + b^2 + c^2 = 1$.

3. **On every such line,** P **is surjective.**

Birational Teichmuller moduli space (cont'd)

4. For any point $x \in \mathbb{P}er$, let $NS(x)$ be the corresponding lattice of integer $(1,1)$-classes in $H^2(M)$. Then **the set of hyperkaehler structures can be determined explicitly** from Boucksom's divisorial Zariski decomposition.

5. **The whole Teichmüller space is covered by such curves,** hence $\text{Teich}_b \xrightarrow{P} \mathbb{P}er$ satisfies the following condition:

(*) For each closed subset $V \subset \mathbb{P}er$ with smooth boundary, and each connected component $W \subset P^{-1}(V)$, **the restriction $W \xrightarrow{P} V$ is surjective.**

6. This condition (*) **always implies that P is a covering.**

Divisorial Zariski decomposition

DEFINITION: A class $\eta \in H^{1,1}(M)$ is called **pseudoeffective** if it can be represented by a positive current, and **nef** if it lies in the closure of a Kaehler cone.

DEFINITION: A **modified nef cone** (also "birational nef cone" and "movable nef cone") is a closure of a union of all nef cones for all bimeromorphic models of a holomorphically symplectic manifold M.

THEOREM: (D. Huybrechts, S. Boucksom)
The modified nef cone is dual to the pseudoeffective cone under the Bogomolov-Beauville-Fujiki pairing.

The divisorial Zariski decomposition theorem: (S. Boucksom)
Let M be a simple hyperkähler manifold. Then **every pseudoeffective class can be decomposed as a sum $\eta = \nu + \sum_i a_i[E_i]$**, where ν is modified nef, a_i positive numbers, and E_i exceptional divisors satisfying $q(E_i, E_i) < 0$.

COROLLARY: Let M be a hyperkaehler manifold with $H^{1,1}(M, \mathbb{Z}) = 0$. **Then every pseudoeffective class is modified nef.**

Proof: Indeed, on such M **there are no exceptional divisors.** ■

A Kähler cone for generic hyperkähler manifolds

COROLLARY: Let M be a hyperkaehler manifold with $H^{1,1}(M,\mathbb{Z}) = 0$. **Then the modified nef cone is self-dual.**

REMARK: For any nef classes ν, ν', one has $q(\nu,\nu') \geqslant 0$. Therefore, **the nef cone is contained in its dual.** Moreover, the nef cone K_n is contained in one of two components K^+ of a set $\{\nu \in H^{1,1}(M) \mid q(\nu,\nu) \geqslant 0\}$, and therefore, **the dual nef cone contains K^+:**

$$K_n \subset K^+ \subset K_n^*$$

REMARK: For $H^{1,1}(M,\mathbb{Z}) = 0$, the modified nef cone K_{mn} is self-dual, but all elements of K_{mn} satisfy $q(\nu,\nu) \geqslant 0$. This gives

$$K_{mn} \subset K^+ \subset K_{mn}^* = K_{mn} \subset K^+.$$

We obtain

COROLLARY: Let M be a hyperkaehler manifold with $H^{1,1}(M,\mathbb{Z}) = 0$. **Then the Kaehler cone is one of two components of the set $\{\nu \in H^{1,1}(M) \mid q(\nu,\nu) \geqslant 0\}$.** ■