Global Torelli theorem for hyperkähler manifolds

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Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2 y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is **skew-symmetric.**

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The almost complex structure I **is integrable**, and the Hermitian form ω is **closed**.

(ii) **One has $\nabla(I) = 0$,** where ∇ is the Levi-Civita connection

$$\nabla : \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M.

Hyperkähler manifolds

DEFINITION: (Calabi, 1978) A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

REMARK: A hyperkähler manifold **has three symplectic forms** $\omega_I := g(I\cdot,\cdot)$, $\omega_J := g(J\cdot,\cdot)$, $\omega_K := g(K\cdot,\cdot)$.

REMARK: $\omega_J + \sqrt{-1}\omega_K$ is a holomorphic symplectic form on (M, I). Therefore, **any hyperkähler manifold is also holomorphically symplectic.**

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

DEFINITION: For the rest of this talk, **a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.**

DEFINITION: A hyperkähler manifold M is called **simple** if $H^1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

Bogomolov's decomposition: Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

Further on, all hyperkähler manifolds are assumed to be simple.

EXAMPLES.

EXAMPLE: An even-dimensional complex torus.

EXAMPLE: A non-compact example: $T^*\mathbb{C}P^n$ (Calabi).

REMARK: $T^*\mathbb{C}P^1$ **is a resolution of a singularity** $\mathbb{C}^2/\pm 1$**.**

REMARK: Let M be a 2-dimensional complex manifold with holomorphic symplectic form outside of singularities, which are all of form $\mathbb{C}^2/\pm 1$. Then **its resolution is also holomorphically symplectic.**

EXAMPLE: Take a 2-dimensional complex torus T, then all the singularities of $T/\pm 1$ are of this form. Its resolution $\widetilde{T/\pm 1}$ is called **a Kummer surface**. **It is holomorphically symplectic**. A **K3 surface** is a deformation of a Kummer surface.

EXAMPLE: There exist hyperkähler resolutions of singularities of symmetric powers of a K3 surface and a torus **("Hilbert schemes of K3", "generalized Kummer varieties")**.

The Teichmüller space and the mapping class group

Definition: Let M be a compact complex manifold, and $\operatorname{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by Teich the space of complex structures on M, and let $\operatorname{Teich} := \operatorname{Teich}/\operatorname{Diff}_0(M)$. We call it **the Teichmüller space.**

Remark: Teich is **a finite-dimensional complex space** (Kodaira), but often **non-Hausdorff**.

Definition: Let $\operatorname{Diff}_+(M)$ be the group of oriented diffeomorphisms of M. We call $\Gamma := \operatorname{Diff}_+(M)/\operatorname{Diff}_0(M)$ **the mapping class group**. The **coarse moduli space of complex structures on M** is a connected component of $\operatorname{Teich}/\Gamma$.

THEOREM: (Kodaira) **A small deformation of a compact Kähler manifold is again Kähler.**

COROLLARY: Small deformations of hyperkähler manifolds are hyperkähler.

REMARK: For hyperkähler manifolds, it is convenient to take for Teich **the space of all complex structures of hyperkähler type**, that is, **holomorphically symplectic and Kähler**. It is open in the usual Teichmüller space.

The Bogomolov-Beauville-Fujiki form

THEOREM: (Fujiki). Let $\eta \in H^2(M)$, and $\dim M = 2n$, where M is hyperkähler. **Then $\int_M \eta^{2n} = cq(\eta,\eta)^n$, for some quadratic form q on $H^2(M)$.**

DEFINITION: This form is called **Bogomolov-Beauville-Fujiki form** (BBF form). **It is defined by this relation uniquely, up to a sign.** The sign is determined from the following formula (Bogomolov, Beauville)

$$\lambda q(\eta,\eta) = (n/2)\int_X \eta\wedge\eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n-1} - \\ -(1-n)\left(\int_X \eta\wedge\Omega^{n-1}\wedge\overline{\Omega}^{n}\right)\left(\int_X \eta\wedge\Omega^{n}\wedge\overline{\Omega}^{n-1}\right)$$

where Ω is the holomorphic symplectic form, and $\lambda = C_{2n-2}^{n-1}\int_M \wedge\Omega^n\wedge\overline{\Omega}^n > 0$.

CLAIM: **q has signature $(b_2-3,3)$.**

CLAIM: The Bogomolov-Beauville-Fujiki form **is preserved by diffeomorphisms of M.**

REMARK: For a K3 surface, the BBF form is equal to the Poincare pairing.

Computation of the mapping class group

THEOREM: (Sullivan) Let M be a compact simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geqslant 3$. Denote by Γ the group of automorphisms of an algebra $H^*(M,\mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then **the natural map** $\operatorname{Diff}_+(M)/\operatorname{Diff}_0 \longrightarrow \Gamma$ **has finite kernel, and its image has finite index in Γ.**

THEOREM: Let M be a simple hyperkähler manifold, and Γ as above. Then
(i) $\Gamma\Big|_{H^2(M,\mathbb{Q})}$ **is an arithmetic subgroup** of $O(H^2(M,\mathbb{Q}),q)$.
(ii) The map $\Gamma \longrightarrow O(H^2(M,\mathbb{Q}),q)$ **has finite kernel.**

REMARK: An **arithmetic group** is a subgroup of finite index in an algebraic Lie group over integers, such as $O(\mathbb{Z}^n,q)$.

THEOREM: (Huybrechts) **There are only finitely many connected components** of Teich.

COROLLARY: The subgroup Γ_I of mapping class group preserving a given connected component of Torelli space **is also arithmetic.**

The period map

Remark: For any $J \in$ Teich, (M,J) is also a simple hyperkähler manifold, hence $H^{2,0}(M,J)$ is one-dimensional.

Definition: Let $P:$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ map J to a line $H^{2,0}(M,J) \in \mathbb{P}H^2(M,\mathbb{C})$. The map $P:$ Teich $\longrightarrow \mathbb{P}H^2(M,\mathbb{C})$ is called **the period map**.

REMARK: P **maps** Teich **into an open subset of a quadric,** defined by

$$\mathbb{P}er := \{l \in \mathbb{P}H^2(M,\mathbb{C}) \ \mid \ q(l,l) = 0, q(l,\bar{l}) > 0.$$

It is called **the period space** of M.

REMARK: $\mathbb{P}er = SO(b_2-3,3)/SO(2) \times SO(b_2-3,1)$

BOGOMOLOV'S THEOREM: Let M be a simple hyperkähler manifold, and Teich its Teichmüller space. Then **the period map** $P:$ Teich $\longrightarrow \mathbb{P}er$ **is locally a diffeomorphism** (in algebraic geometry, a local diffeomorphism is called **an etale map**).

REMARK: Bogomolov's theorem implies that Teich **is smooth.** It is **non-Hausdorff** even in the simplest examples.

Hausdorff reduction

REMARK: A non-Hausdorff manifold is a topological space locally diffeomorphic to $\mathbb{R}^n$.

DEFINITION: Let M be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

DEFINITION: Let X be a topological space. Suppose that $\sim$ is an equivalence relation, and the quotient of $M/\sim$ is Hausdorff. Then $M/\sim$ is called **a Hausdorff reduction** of M.

THEOREM: Let Teich be a Teichmüller space of a hyperkaehler manifold. **Then it admits a Hausdorff reduction** which is a smooth complex manifold.

THEOREM: (D. Huybrechts) If I_1, $I_2 \in$ Teich are non-separate points, then $P(I_1) = P(I_2)$, and (M, I_1) **is bimeromorphically equivalent** to (M, I_2).

DEFINITION: The Hausdorff reduction $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **a birational Teichmüller space** of a hyperkähler manifold.

Global Torelli theorem

THEOREM: Let $P: \operatorname{Teich} \longrightarrow \mathbb{P}er$ be a period map of a hyperkähler manifold. **Then P is a Hausdorff reduction,** for each connected component of Teich. In particular, it is surjective and generically one-to-one on each component.

DEFINITION: Define **the birational moduli space** of a hyperkähler manifold as a quotient $\operatorname{Teich}_b/\Gamma$, where Γ is a mapping class group.

THEOREM: Let M be a hyperkähler manifold, and W a connected component of its birational moduli space. **Then W is isomorphic to $\mathbb{P}er/\Gamma_I$, where $\mathbb{P}er = SO(b_2-3,3)/SO(2)\times SO(b_2-3,1)$ and Γ_I is an arithmetic group in $O(b_2-3,3)$**

A CAUTION: Usually "the global Torelli theorem" is understood as a theorem about Hodge structures. For K3 surfaces, **the Hodge structure on $H^2(M,\mathbb{Z})$ determines the complex structure**. For $\dim_{\mathbb{C}} M > 2$, **it is false**, except for a few cases when it follows from the above theorem and an explicit description of Γ due to E. Markman.

THE END!

The Hodge-theoretic Torelli theorem

REMARK: The group $O(p,q)$ $(p,q > 0)$ has **4 connected components**, corresponding to the orientations of positive p-dimensional and negative q-dimensional planes.

DEFINITION: **A spin-orientation** on a space with signature p,q is a choice of orientation of positive p-dimensional planes. The group preserving spin-orientation is denoted by $O^+(p,q)$.

REMARK: **The Hodge-theoretic birational Torelli theorem** is an isomorphism $\operatorname{Teich}/\Gamma_I \longrightarrow \mathbb{P}er/O^+(H^2(M,\mathbb{Z}),q)$.

REMARK: A birational Hodge-theoretic Torelli theorem is true if and only if Γ_I (the stabilizer of a Torelli component in the mapping class group) **is isomorphic to $O^+(H^2(M,\mathbb{Z}),q)$.**

THEOREM: (Markman) The for $M = K3^{[n+1]}$, the group Γ_I **is a subgroup of $O^+(H^2(M,\mathbb{Z}),q)$ generated by spin-oriented reflections.**

THEOREM: Let $M = K3^{[n+1]}$ with n a prime power. Then the (usual) global Torelli theorem holds birationally: **two deformations of a Hilbert scheme with isomorphic Hodge structures are bimeromorphic**. **For other n, it is false** (Markman).

Weakly Hausdorff manifolds

DEFINITION: A point $x \in X$ is called **Hausdorff** if $x \not\sim y$ for any $y \neq x$.

DEFINITION: Let M be an n-dimensional real analytic manifold, not necessarily Hausdoff. Suppose that **the set $Z \subset M$ of non-Hausdorff points is contained in a countable union of real analytic subvarieties** of codim $\geqslant 2$. Suppose, moreover, that

(S) For every $x \in M$, there is a closed neighbourhood $B \subset M$ of x and a continuous surjective map $\Psi : B \longrightarrow \mathbb{R}^n$ to a closed ball in $\mathbb{R}^n$, **inducing a homeomorphism** on an open neighbourhood of x.

Then M is called **a weakly Hausdorff manifold**.

REMARK: The period map satisfies (S). Also, the non-Hausdorff points of Teich **are contained in the countable union of divisors.**

THEOREM: A **weakly Hausdorff manifold X admits a Hausdorff reduction.** In other words, the quotient $X/\sim$ is a Hausdorff. Moreover, $X \longrightarrow X/\sim$ is locally a homeomorphism.

Birational Teichmüller moduli space

DEFINITION: The space $\operatorname{Teich}_b := \operatorname{Teich}/\sim$ is called **the birational Teichmüller space** of M.

THEOREM: The period map $\operatorname{Teich}_b \xrightarrow{P} \mathbb{P}er$ is an isomorphism.

Idea of a proof:

0. Since $\mathbb{P}er = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ **is simply connected, it will suffice to show it's a covering.**

1. **It is etale** (Bogomolov).

2. For each hyperkähler structure (I, J, K) on M, **there is a whole S^2 of complex structures** $L = aI + bJ + cK$ on M, for $a^2 + b^2 + c^2 = 1$.

3. **On every such line, P is surjective.**

Birational Teichmüller moduli space (cont'd)

4. For any point $x \in \mathbb{P}er$, let $NS(x)$ be the corresponding lattice of integer $(1,1)$-classes in $H^2(M)$. Then **the set of Kaehler classes can be determined explicitly** when $\operatorname{rk} NS(x) \leqslant 1$. Every Kaehler class gives a hyperkaehler structure, hence a line in Teich_b.

5. **The whole Teichmüller space is covered by such curves,** hence $\operatorname{Teich}_b \xrightarrow{P} \mathbb{P}er$ satisfies the following condition:

(*) For each closed subset $V \subset \mathbb{P}er$ with smooth boundary, and each connected component $W \subset P^{-1}(V)$, **the restriction $W \xrightarrow{P} V$ is surjective.**

6. This condition (*) **always implies that P is a covering.**