Transcendental Hodge algebra

Misha Verbitsky

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Hodge structures

DEFINITION: Let $V_{\mathbb R}$ be a real vector space. **A (real) Hodge structure of weight** w on a vector space $V_{\mathbb C} = V_{\mathbb R} \otimes_{\mathbb R} \mathbb C$ is a decomposition $V_{\mathbb C} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called **rational Hodge structure** if one fixes a rational lattice $V_{\mathbb Q}$ such that $V_{\mathbb R} = V_{\mathbb Q} \otimes \mathbb R$. A Hodge structure is equipped with $U(1)$-action, with $u \in U(1)$ acting as u^{p-q} on $V^{p,q}$. **Morphism** of Hodge structures is a rational map which is $U(1)$-invariant.

DEFINITION: Polarization on a rational Hodge structrure of weight w is a $U(1)$-invariant non-degenerate 2-form $h \in V_{\mathbb Q}^* \otimes V_{\mathbb Q}^*$ (symmetric or antisymmetric depending on parity of w) which satisfies

$$-\sqrt{-1}^{\,p-q} h(x, \overline{x}) > 0 \qquad (*)$$

("Riemann-Hodge relations") for each non-zero $x \in V^{p,q}$.

Hodge structures (2)

DEFINITION: A **simple object** of an abelian category is an object which has no proper subobjects. An abelian category is **semisimple** if any object is a direct sum of simple objects.

CLAIM: Category of polarized Hodge structures in semisimple.

Proof: Orthogonal complement of a Hodge substructure $V' \subset V$ with respect to h is again a Hodge substructure, and this complement does not intersect V'; both assertions follow from the Riemann-Hodge relations. ■

EXAMPLE: Let (M, ω) be a compact Kähler manifold. Then the Hodge decomposition $H^w(M, \mathbb{C}) = \oplus H^{p,q}(M)$ defines a Hodge structure on $H^w(M, \mathbb{C})$. If we restrict ourselves to the primitive cohomology space

$$H^w_{\text{prim}} := \{\eta \in H^w(M) \quad (*\eta) \wedge \omega = 0\},$$

and consider $h(x,y) := \int_M x \wedge y \wedge \omega^{\dim_{\mathbb{C}} M - w}$, relations (*) become the usual Hodge-Riemann relations. If, in addition, the cohomology class of ω is rational (in this case, by Kodaira theorem, (M, ω) is projective) the space $H^w_{\text{prim}}(M)$ is also rational, and **the Hodge decomposition $H^w_{\text{prim}}(M, \mathbb{C}) = \oplus H^{p,q}_{\text{prim}}(M)$ defines a polarized, rational Hodge structure.**

Mumford-Tate group

DEFINITION: Let V be a Hodge structure over $\mathbb{Q}$, and ρ the corresponding $U(1)$-action. **Mumford-Tate group** (Mumford, 1966; Mumford called it "the Hodge group") is the smallest algebraic group over $\mathbb{Q}$ containing ρ.

THEOREM: Let V be a rational, polarized Hodge structure, and $\mathsf{MT}(V)$ its Mumford-Tate group. Consider the tensor algebra of V, $W = T^{\otimes}(V)$ with the Hodge structure (also polarized) induced from V. Let W_h be the space of all ρ-invariant rational vectors in W (such vectors are called **"Hodge vectors"**). **Then $\mathsf{MT}(V)$ coincides with the stabilizer**

$$\mathsf{St}_{GL(V)}(W_h) := \{g \in GL(V) \ \ | \ \ \forall w \in W_h, g(w) = w\}.$$

Proof: Follows from the Chevalley's theorem on tensor invariants. ∎

Corollary 1: Let $V_{\mathbb{Q}}$ be a rational Hodge structure, and $W \subset V_{\mathbb{C}}$ a subspace. **Then the following are equivalent.**
(i) W is a Hodge substructure.
(ii) W is Mumford-Tate invariant. ∎

Mumford-Tate group and the $\mathcal{G}al(\mathbb{C}/\mathbb{Q})$-action

DEFINITION: Let $V_{\mathbb{Q}}$ be a $\mathbb{Q}$-vector space, and $V_{\mathbb{C}} := V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ its complexification, equipped with a natural Galois group $\mathcal{G}al(\mathbb{C}/\mathbb{Q})$ action. We call a complex subspace $T \subset V_{\mathbb{C}}$ **rational** if $T = T_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$, where $T_Q = V_{\mathbb{Q}} \cap T$.

REMARK: A complex subspace $W \subset V_{\mathbb{C}}$ is rational if and only if it is $\mathcal{G}al(\mathbb{C}/\mathbb{Q})$-invariant.

This implies

Claim 1: Consider a subspace $W \subset V_{\mathbb{C}}$, and let $\tilde{W}_{\mathbb{Q}} \subset V_{\mathbb{Q}}$ be a smallest subspace of $V_{\mathbb{Q}}$ such that $\tilde{W}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} \supset W$. **Then $W_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ is generated by $\sigma(W)$, for all $\sigma \in \mathcal{G}al(\mathbb{C}/\mathbb{Q})$.** ■

PROPOSITION: Let $V_{\mathbb{Q}}$ be a rational, polarized Hodge structure, ρ the corresponding $U(1)$-action, and MT its Mumford-Tate group. **Then MT is a Zariski closure of a group generated by $\sigma(\rho)$, for all $\sigma \in \mathcal{G}al(\mathbb{C}/\mathbb{Q})$.**

Proof: Since MT is rational, it is preserved by $\mathcal{G}al(\mathbb{C}/\mathbb{Q})$. The algebraic closure of the group generated by $\sigma(\rho)$ coincides with MT, because it is a smallest group which is rational, Zariski closed and contains ρ. ■

Transcendental Hodge lattice

THEOREM: Let $V_{\mathbb{Q}}$ be a rational, polarized Hodge structure of weight w, $V_C = \bigoplus_{\substack{p+q=w \\ p,q\geqslant 0}} V^{p,q}$, and $V^{tr} \subset V_{\mathbb{C}}$ a minimal rational subspace containing $V^{d,0}$. **Then V^{tr} is a Hodge substructure.**

Proof: By Claim 1, V^{tr} is generated by $\sigma(V^{d,0})$ for all $\sigma \in \mathcal{Gal}(\mathbb{C}/\mathbb{Q})$. Since $V^{d,0}$ is preserved by the $U(1)$-action ρ, V^{tr} is preserved by the group generated by all $\sigma(\rho)$, which is MT. Finally, a rational subspace is a Hodge substructure if and only if it MT-invariant (Corollary 1). ■

THEOREM: **Transcendental Hodge lattice is a birational invariant.**

Proof: Let $\varphi: \ X \longrightarrow Y$ be a birational morphism of projective varieties. Then $\varphi^*: \ H^d(Y) \longrightarrow H^d(X)$ induces isomorphism on $H^{d,0}$. Therefore, it is injective on $H^d_{tr}(Y)$. Indeed, its kernel is a Hodge substructure of $H^d_{tr}(Y)$ not intersecting $H^{d,0}$, which is impossible. Applying the same argument to the dual map, we obtain that φ^* is also surjective on $H^d_{tr}(Y)$. ■

Transcendental Hodge algebra

PROPOSITION: Let M be a projective Kähler manifold, $H^*_{tr}(M) := \oplus_d H^d_{tr}(M)$ the direct sum of all transcendental Hodge lattices, and $H^*_{tr}(M)^\perp$ its orthogonal complement with respect to the polarization form $h(x,y) := \int_M x \wedge y \wedge \omega^{\dim_{\mathbb{C}} M - w}$. **Then $H^*_{tr}(M)^\perp$ is an ideal in the cohomology algebra.**

Proof: The space $V^* := H^*_{tr}(M)^\perp$ is a maximal Hodge structure contained in

$$A^* := \bigoplus_{\substack{p+q=w \\ p,q>0}} V^{p,q}.$$

The space A^* is clearly an ideal. For any two Hodge substructures $X, Y \subset H^*(M)$, the product $X \cdot Y$ also rational and $U(1)$-invariant, hence it is also a Hodge substructure. However, A^* is an ideal, hence $X \cdot V^*$ is a Hodge structure contained in A^*. Therefore, $H^*(M) \cdot V^*$ is contained in V^*. ■

DEFINITION: The quotient algebra $H^*(M)/H^*_{tr}(M)^\perp = H^*_{tr}(M)$ is called **the transcendental Hodge algebra** of M.

PROPOSITION:
Transcendental Hodge algebra is a birational invariant.

Proof: Same as for transcendental Hodge lattices. ■

Zarhin's results about Hodge structures of K3 type

DEFINITION: A polarized, rational Hodge structure $V_{\mathbb{C}} = \bigoplus_{\substack{p+q=2\\ p,q\geqslant 0}} V^{p,q}$ of weight 2 with $\dim V^{2,0} = 1$ is called **a Hodge structure of K3 type**.

DEFINITION: A Hodge structure is called **simple** if it has no proper Hodge substructures.

REMARK: Since the category of polarized Hodge structures is semisimple, **every Hodge structure is a direct sum of simple ones.**

REMARK: Let M be a projective K3 surface, and $V_{\mathbb{Q}}$ its transcendental Hodge lattice. **Then in is simple and of K3 type.**

PROPOSITION: (Zarhin) Let $V_{\mathbb{Q}}$ be a simple Hodge structure of K3 sype, and $E = \operatorname{End}(V_{\mathbb{Q}})$ an algebra of its endomorphisms in the category of Hodge structures. **Then E is a number field.**

Proof: By Schur's lemma, E has no zero divisors, hence it is a division algebra. Since $E \subset \operatorname{End}(V_{\mathbb{Q}})$, it is countable. **To prove that it is a number field, it remains to show that E is commutative.** However, E acts on a 1-dimensional space $V^{2,0}$. This defines a homomorphism from E to $\mathbb{C}$, which is injective, because E is a division algebra. ■

Zarhin's results about Hodge structures of K3 type (2)

THEOREM: (Zarhin) Let $V_{\mathbb{Q}}$ be a Hodge structure of K3 type, and $E := \operatorname{End}(V_{\mathbb{Q}})$ its endomorphism field. **Then E is either totally real (that is, all its embeddings to $\mathbb{C}$ are real) or is an imaginary quadratic extension of a totally real field E_0.**

Proof. Step 1: Let $a \in E$ be an endomorphism, and a^* its conjugate with respect to the polarization h. Since the polarization is rational and $U(1)$-invariant, the map a^* also preserves the Hodge decomposition. Then $a^* \in E$. Denote the generator of $V^{2,0}$ by Ω. Then $h(a(\Omega), a(\overline{\Omega})) = h(a^*a(\Omega), \overline{\Omega}) > 0$, hence $\frac{a^*a(\Omega)}{\Omega}$ is a positive real number. Then, the embedding $E \hookrightarrow \mathbb{C}$ induced by $b \longrightarrow \frac{b(\Omega)}{\Omega}$ maps a^*a to a positive real number.

Step 2: Since the group $\mathcal{G}al(\mathbb{C}/\mathbb{Q})$ acts on V by automorphisms, it preserves E, hence $[E : \mathbb{Q}]$ is a Galois extension. This means that $\mathcal{G}al(\mathbb{C}/\mathbb{Q})$ acts transitively on embeddings from E to $\mathbb{C}$. Therefore, **all embeddings $E \hookrightarrow \mathbb{C}$ map a^*a to a positive real number.**

Zarhin's results about Hodge structures of K3 type (3)

THEOREM: (Zarhin) Let $V_{\mathbb{Q}}$ be a Hodge structure of K3 type, and $E := \operatorname{End}(V_{\mathbb{Q}})$ its endomorphism field. **Then E is either totally real (that is, all its embeddings to $\mathbb{C}$ are real) or is an imaginary quadratic extension of a totally real field E_0.**

Step 3: The map $a \longrightarrow a^*$ is either a non-trivial involution or identity. In the second case, all embeddings $E \hookrightarrow \mathbb{C}$ map a^2 to a positive real number, and E is totally real. In the second case, $\tau(a) = a^*$ is an involution, hence its fixed set $E^\tau =: E_0$ is a degree 2 subfield of E, with τ the generator of the Galois group $\mathcal{Gal}(E/E_0)$.

Step 4: Let r be the root of the corresponding quadratic equation $x^2 - u = 0$. Then $\tau(r) = -r$, which gives $-r^2 = r\tau(r) > 0$, and $-r^2$ is positive and real for all embeddings $E \hookrightarrow \mathbb{C}$. Therefore, $[E : E_0]$ is an imaginary quadratic extension. ■

Zarhin's results about Mumford-Tate group

THEOREM: (Zarhin) Let $V_{\mathbb{Q}}$ be a Hodge structure of K3 type, and $E :=$ End$(V_{\mathbb{Q}})$ the corresponding number field. Denote by $SO_E(V)$ the group of E-linear isometries of V for $[E:\mathbb{Q}]$ totally real, and by $U_E(V)$ the group of E-linear isometries of V for E an imaginary quadratic extension of a totally real field. **Then the Mumford-Tate group** MT **is $SO_E(V)$ in the first case, and $U_E(V)$ in the second.**

Proof: *Zarhin, Yu.G.,* Hodge groups of K3 surfaces, *Journal für die reine und angewandte Mathematik Volume 341, page 193-220, 1983.* ■

REMARK: As a Lie group, $SO_E(V)$ is $SO(\mathbb{C}^n)$, where n is dimension of $V_{\mathbb{Q}}$ over E, because h is E-linear for the totally real E. For an imaginary quadratic case, $U_E(V)$ is $U(C^n)$, because h is E_0-linear and Hermitian with respect to the complex conjugation map in $\mathcal{G}al(E/E_0)$.

Irreducible representations of $SO(V)$

Claim 2: Let V be a vector space equipped with a non-degenerate symmetric product h, and $b \in \operatorname{Sym}^2(V)$ an $SO(V)$-invariant bivector dual to h. **Denote by** $\operatorname{Sym}^*_+(V)$ the quotient of $\operatorname{Sym}^*(V)$ by the ideal generated by b. **Then** $\operatorname{Sym}^i_+(V)$ **is irreducible as a representation of $SO(V)$.**

Proof: It is well known (see [Weyl]) that $\operatorname{Sym}^i(V)$ is decomposed to a direct sum of irreducible representations of $SO(V)$ as follows:

$$\operatorname{Sym}^i(V) = \operatorname{Sym}^i_i(V) \oplus \operatorname{Sym}^i_{i-2}(V) \oplus ... \oplus \operatorname{Sym}^i_{i-2\lfloor i/2 \rfloor}(V),$$

where $\operatorname{Sym}^i_{i-k}(V) \cong \operatorname{Sym}^{i-k}_{i-k}(V)$ and the subrepresentation $\operatorname{Sym}^i_{i-2}(V) \oplus ... \oplus \operatorname{Sym}^i_{i-2\lfloor i/2 \rfloor}(V)$ is the image of $\operatorname{Sym}^{i-2}(V)$ under the map $\alpha \longrightarrow \alpha \cdot b$. Then $\operatorname{Sym}^i_+(V) = \operatorname{Sym}^i_i(V)$, hence irreducible. ■

Transcendental Hodge algebra for hyperkähler manifolds

Theorem 0: (V., 1995) Let M be a maximal holonomy hyperkähler manifold, $\dim_{\mathbb{C}} M = 2n$, and $H^*_{(2)}(M)$ subalgebra in cohomology generated by $H^2(M)$. Then $H^{2i}_{(2)}(M) = \operatorname{Sym}^i H^2(M)$ for all $i \leqslant n$.

The main result of today's talk:

THEOREM: Let M be a projective maximal holonomy hyperkähler manifold, $\dim_{\mathbb{C}} M = 2n$, $H^{2*}_{tr}(M)$ its transcendental Hodge algebra, and $E = \operatorname{End}(V)$ the number field of endomorphisms of its transcendental lattice $V = H^2_{tr}(M)$. Let $\operatorname{Sym}(V)_E$ denote the E-linear symmetric product. **Then $H^{2*}_{tr}(M) = \bigoplus_{i=0}^n \operatorname{Sym}^i(V)_E$ for E imaginary quadratic extension, and $H^{2*}_{tr}(M) = \bigoplus_{i=0}^n \operatorname{Sym}^i_+(V)_E$ for E totally real.**

Proof: By definition, $H^{2i}_{tr}(M)$ is the smallest Hodge substructure (that is, the Mumford-Tate subrepresentation) of $H^{2i}(M)$ containing Ω^i. By Theorem 0, one could replace $H^{2i}(M)$ with $\operatorname{Sym}^i(V)$. This means that we need to find the smallest $U_E(V)$- and $SO_E(V)$-representation containing Ω^i. For V a fundamental representation of $U(V)$, the space $\operatorname{Sym}^i(V)$ is irreducible, and for $SO(V)$, the irreducible component containing Ω^i is precisely $\operatorname{Sym}^i_+(V)$ (Claim 2). ■

Non-degeneracy of the map $x \longrightarrow x^n$ in $H^*_{tr}(M)$

THEOREM: Let M be a projective maximal holonomy hyperkähler manifold, $\dim_{\mathbb{C}} M = 2n$ and $x \in H^2_{tr}(M)$ a non-zero vector. **Then $x^n \neq 0$ in $H^n_{tr}(M)$.**

Proof: When $H^*_{tr}(M) = \mathrm{Sym}^*(V)$, this is trivial. When $H^*_{tr}(M) = \mathrm{Sym}^*_+(V)$, this is proven as follows. The action of $SO(V)$ on the projectivization $\mathbb{P}V$ has only two orbits: the quadric $Q := \{x \mid h(x,x) = 0\}$ and the rest. The map $P(x) = x^n$ is non-zero, hence it is non-zero on the open orbit. To check that it is non-zero on the quadric, notice that Ω belongs to Q, and Ω^n is non-zero in $\mathrm{Sym}^n_+(V)$. ■

DEFINITION: Let V be a $4n$-dimensional vector space, and $\Psi : W \longrightarrow \Lambda^2(V)$ a linear map. Assume that $\Psi(\omega)$ is a symplectic form for general $\omega \in W$, and has rank $\frac{1}{2}\dim W$ for ω in a non-degenerate quadric $Q \subset W$. Then Ψ is called **k-symplectic structure on V**, where $k = \dim W$.

THEOREM: (Soldatenkov-V.) Let V be a k-symplectic space. **Then V is a Clifford module over a Clifford algebra $\mathcal{C}l(W_0)$, with $\dim W_0 = \dim W - 1 = k$, and $\dim V$ divisible by $2^{\lfloor (k-1)/2 \rfloor}$.**

Application to symplectically embedded tori

Corollary 1: Let M be a projective maximal holonomy hyperkähler manifold, $N \hookrightarrow M$ a torus, $\dim_{\mathbb{C}} N = 2n$, embedded holomorphically symplectically in M, and $x \in H^2(N)$ a restriction of a non-zero class $x \in H^2_{tr}(M)$. **Then $x^n \neq 0$.**

Proof: Since the embedding $N \hookrightarrow M$ is holomorphically symplectic, the restriction map $H^{2n}_{tr}(M) \longrightarrow H^{2n}_{tr}(N)$ is non-zero, hence injective. On the other hand, $x^n \neq 0$ in $H^{2n}_{tr}(M)$ as shown above. ■

COROLLARY: Let M be a projective hyperkähler manifold with Picard rank 1, generic in its deformation space, and $N \hookrightarrow M$ a holomorphic symplectic torus. **Then $\dim N$ is divisible by $2^{\lfloor (b_2(M)-2)/2 \rfloor}$.**

Proof: By Corollary 1, $H^1(T)$ is k-symplectic, where $k = \dim H^2_{tr}(M) = b_2(M) - 1$. Then it is a Clifford module over $\mathcal{C}l(k-1)$ by Soldatenkov-V. ■