

Twistor transform, instantons and rational curves

Misha Verbitsky

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Plan

1. Hyperkähler and quaternionic-Kähler manifolds and their twistor spaces
2. Chern connection
3. Hyperholomorphic bundles and twistor transform
4. Twistor transform for mathematical instantons

Hyperkähler manifolds

DEFINITION: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K , satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K .

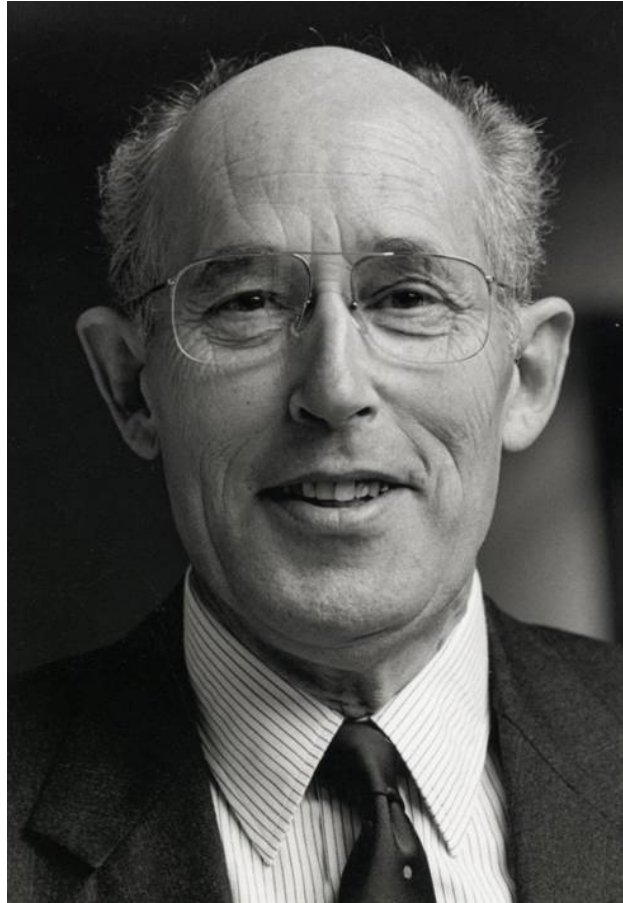
REMARK: A hyperkähler manifold **has three symplectic forms**

$$\omega_I := g(I\cdot, \cdot), \quad \omega_J := g(J\cdot, \cdot), \quad \omega_K := g(K\cdot, \cdot).$$

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the Levi-Civita connection preserves I, J, K .

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_x M)$ generated by parallel translations (along all paths) is called **the holonomy group** of M .

REMARK: A hyperkähler manifold can be defined as a manifold which **has holonomy in $Sp(n)$** (the group of all endomorphisms preserving I, J, K).



Marcel Berger

Classification of holonomies

THEOREM: (de Rham) A complete, simply connected Riemannian manifold with non-irreducible holonomy **splits as a Riemannian product.**

THEOREM: (Berger's theorem, 1955) Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
<i>Holonomy</i>	<i>Geometry</i>
$SO(n)$ acting on \mathbb{R}^n	Riemannian manifolds
$U(n)$ acting on \mathbb{R}^{2n}	Kähler manifolds
$SU(n)$ acting on \mathbb{R}^{2n} , $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on \mathbb{R}^{4n}	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on \mathbb{R}^{4n} , $n > 1$	quaternionic-Kähler manifolds
G_2 acting on \mathbb{R}^7	G_2 -manifolds
$Spin(7)$ acting on \mathbb{R}^8	$Spin(7)$ -manifolds

Quaternionic-Kähler manifolds

DEFINITION: A **quaternionic-Kähler manifold** is a Riemannian (M, g) manifold with holonomy in $Sp(n) \times Sp(1)/\{\pm 1\}$. Equivalently, it is a Riemannian manifold **equipped with a 3-dimensional sub-bundle** $E \subset \mathfrak{so}(TM)$ satisfying the following

1. E is closed with respect to the commutator, and isomorphic to $\mathfrak{so}(3)$ acting as imaginary quaternions at each point of M
2. $\nabla E \subset E \otimes \Lambda^1 M$.

REMARK: A quaternionic-Kähler manifold is **Einstein**, that is, **satisfies** $\text{Ric}(M) = \lambda g$, for some constant $\lambda \in \mathbb{R}$ (here, $\text{Ric}(M) \in \text{Sym}^2 T^*M$ is a Ricci curvature).

REMARK: Whenever the constant λ is equal 0, M is hyperkähler, otherwise it's **not hyperkähler**. Even if **hyperkähler manifolds are always quaternionic-Kähler**, when people say “quaternionic-Kähler” they actually mean “quaternionic-Kähler with $\lambda \neq 0$.”

Further on, all quaternionic-Kähler manifolds will be non-Kähler.

Twistor spaces

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$

They are usually non-algebraic. Indeed, if M is compact, for generic a, b, c , (M, L) has no divisors (Fujiki).

DEFINITION: A **twistor space** $\text{Tw}(M)$ of a hyperkähler manifold is a **complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\text{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \rightarrow T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\text{Tw}} = I_m \oplus I_J : T_x \text{Tw}(M) \rightarrow T_x \text{Tw}(M)$ satisfies $I_{\text{Tw}}^2 = -\text{Id}$. **It defines an almost complex structure on $\text{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

EXAMPLE: If $M = \mathbb{H}^n$, $\text{Tw}(M) = \text{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \setminus \mathbb{C}P^{2n-1}$

REMARK: For M compact, $\text{Tw}(M)$ never admits a Kähler structure.

Twistor spaces for quaternionic-Kähler manifolds

DEFINITION: A **twistor space** $\text{Tw}(M)$ of a quaternionic-Kähler manifold (M, g, E) is a total space of a unit sphere bundle on E , equipped with a complex structure as above.

EXAMPLE: If $M = \mathbb{H}P^n$, then $\text{Tw}(M) = \mathbb{C}P^{2n+1}$. In particular, $\text{Tw}(S^4) = \mathbb{C}P^3$.

REMARK: Consider a compact quaternionic-Kähler manifold (M, g) with $\text{Ric}(M) = \lambda g$, $\lambda > 0$. Then $\text{Tw}(M)$ is a **holomorphically contact Fano manifold**. Conversely, **any Kähler-Einstein holomorphically contact Fano manifold is a twistor space of a compact quaternionic-Kähler manifold (M, g) with $\text{Ric}(M) = \lambda g$, $\lambda > 0$.**

One can say that **hyperkähler geometry is holomorphic symplectic geometry, and quaternionic-Kähler is holomorphic contact geometry**

A holomorphic structure operator

DEFINITION: Let $d = d^{0,1} + d^{1,0}$ be the Hodge decomposition of the de Rham differential on a complex manifold, $d^{0,1} : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p,q+1}(M)$ and $d^{1,0} : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+1,q}(M)$. The operators $d^{0,1}$, $d^{1,0}$ are denoted $\bar{\partial}$ and ∂ and called **the Dolbeault differentials**.

REMARK: From $d^2 = 0$, one obtains $\bar{\partial}^2 = 0$ and $\partial^2 = 0$.

REMARK: The operator $\bar{\partial}$ is \mathcal{O}_M -linear.

DEFINITION: Let B be a holomorphic vector bundle, and $\bar{\partial} : B_{C^\infty} \longrightarrow B_{C^\infty} \otimes \Lambda^{0,1}(M)$ an operator mapping $b \otimes f$ to $b \otimes \bar{\partial}f$, where $b \in B$ is a holomorphic section, and f a smooth function. This operator is called **a holomorphic structure operator** on B . **It is correctly defined, because $\bar{\partial}$ is \mathcal{O}_M -linear.**

REMARK: The kernel of $\bar{\partial}$ coincides with the set of holomorphic sections of B .

The $\bar{\partial}$ -operator on vector bundles

DEFINITION: A $\bar{\partial}$ -operator on a smooth bundle is a map $V \xrightarrow{\bar{\partial}} \Lambda^{0,1}(M) \otimes V$, satisfying $\bar{\partial}(fb) = \bar{\partial}(f) \otimes b + f\bar{\partial}(b)$ for all $f \in C^\infty M, b \in V$.

REMARK: A $\bar{\partial}$ -operator on B can be extended to

$$\bar{\partial} : \Lambda^{0,i}(M) \otimes V \longrightarrow \Lambda^{0,i+1}(M) \otimes V,$$

using $\bar{\partial}(\eta \otimes b) = \bar{\partial}(\eta) \otimes b + (-1)^{\tilde{\eta}} \eta \wedge \bar{\partial}(b)$, where $b \in V$ and $\eta \in \Lambda^{0,i}(M)$.

REMARK: If $\bar{\partial}$ is a holomorphic structure operator, then $\bar{\partial}^2 = 0$.

THEOREM: (Atiyah-Bott) Let $\bar{\partial} : V \longrightarrow \Lambda^{0,1}(M) \otimes V$ be a $\bar{\partial}$ -operator, satisfying $\bar{\partial}^2 = 0$. **Then $B := \ker \bar{\partial} \subset V$ is a holomorphic vector bundle of the same rank.**

DEFINITION: $\bar{\partial}$ -operator $\bar{\partial} : V \longrightarrow \Lambda^{0,1}(M) \otimes V$ on a smooth manifold is called a **holomorphic structure operator**, if $\bar{\partial}^2 = 0$.

Connections and holomorphic structure operators

DEFINITION: let (B, ∇) be a smooth bundle with connection and a holomorphic structure $\bar{\partial} : B \rightarrow \Lambda^{0,1}(M) \otimes B$. Consider a Hodge decomposition $\nabla = \nabla^{0,1} + \nabla^{1,0}$,

$$\nabla^{0,1} : B \rightarrow \Lambda^{0,1}(M) \otimes B, \quad \nabla^{1,0} : B \rightarrow \Lambda^{1,0}(M) \otimes B.$$

We say that ∇ is **compatible with the holomorphic structure** if $\nabla^{0,1} = \bar{\partial}$.

DEFINITION: A Chern connection on a holomorphic Hermitian vector bundle is a connection compatible with the holomorphic structure and preserving the metric.

THEOREM: On any holomorphic Hermitian vector bundle, **the Chern connection exists, and is unique.**

REMARK: The curvature of a Chern connection on B is an $\text{End}(B)$ -valued $(1,1)$ -form: $\Theta_B \in \Lambda^{1,1}(\text{End}(B))$.

REMARK: A converse is true, by Atiyah-Bott theorem. Given a Hermitian connection ∇ on a vector bundle B with curvature in $\Lambda^{1,1}(\text{End}(B))$, we obtain a holomorphic structure operator $\bar{\partial} = \nabla^{0,1}$. Then, **∇ is a Chern connection of $(B, \bar{\partial})$.**

Hyperholomorphic connections

REMARK: Let M be a hyperkähler manifold. **The group $SU(2)$ of unitary quaternions acts on $\Lambda^*(M)$ multiplicatively.**

DEFINITION: A **hyperholomorphic connection** on a vector bundle B over M is a Hermitian connection with $SU(2)$ -invariant curvature $\Theta \in \Lambda^2(M) \otimes \text{End}(B)$.

REMARK: Since the invariant 2-forms satisfy $\Lambda^2(M)_{SU(2)} = \bigcap_{I \in \mathbb{C}P^1} \Lambda_I^{1,1}(M)$, **a hyperholomorphic connection defines a holomorphic structure on B for each I induced by quaternions.**

REMARK: Let M be a compact hyperkähler manifold. Then $SU(2)$ preserves harmonic forms, hence **acts on cohomology.**

CLAIM: **All Chern classes of hyperholomorphic bundles are $SU(2)$ -invariant.**

Proof: Use $\Lambda^2(M)_{SU(2)} = \bigcap_{I \in \mathbb{C}P^1} \Lambda_I^{1,1}(M)$. ■

REMARK: **Converse is also true** (for stable bundles). See the next slide.

Kobayashi-Hitchin correspondence

DEFINITION: Let F be a coherent sheaf over an n -dimensional compact Kähler manifold M . Let

$$\text{slope}(F) := \frac{1}{\text{rank}(F)} \int_M \frac{c_1(F) \wedge \omega^{n-1}}{\text{vol}(M)}.$$

A torsion-free sheaf F is called **(Mumford-Takemoto) stable** if for all subsheaves $F' \subset F$ one has $\text{slope}(F') < \text{slope}(F)$. If F is a direct sum of stable sheaves of the same slope, F is called **polystable**.

DEFINITION: A Hermitian metric on a holomorphic vector bundle B is called **Yang-Mills** (Hermitian-Einstein) if the curvature of its Chern connection satisfies $\Theta_B \wedge \omega^{n-1} = \text{slope}(F) \cdot \text{Id}_B \cdot \omega^n$. A Yang-Mills connection is a Chern connection induced by the Yang-Mills metric.

REMARK: Yang-Mills connections minimize the integral

$$\int_M |\Theta_B|^2 \text{Vol}_M$$

Kobayashi-Hitchin correspondence: (Donaldson, Uhlenbeck-Yau). Let B be a holomorphic vector bundle. **Then B admits Yang-Mills connection if and only if B is polystable.** Moreover such a connection is **unique**.

Kobayashi-Hitchin correspondence and hyperholomorphic bundles

CLAIM: Let M be a hyperkähler manifold. Then for any $SU(2)$ -invariant 2-form $\eta \in \Lambda^2(M)$, one has $\eta \wedge \omega^{n-1} = 0$.

COROLLARY: Any hyperholomorphic bundle is Yang-Mills (hence polystable).

REMARK: This implies that a hyperholomorphic connection on a given holomorphic vector bundle is unique (if exists). Such a bundle is called hyperholomorphic.

THEOREM: Let B be a polystable holomorphic bundle on (M, I) , where (M, I, J, K) is hyperkähler. Then the (unique) Yang-Mills connection on B is hyperholomorphic if and only if the cohomology classes $c_1(B)$ and $c_2(B)$ are $SU(2)$ -invariant.

COROLLARY: The moduli space of stable holomorphic vector bundles with $SU(2)$ -invariant $c_1(B)$ and $c_2(B)$ is a hyperkähler manifold.

COROLLARY: Let (M, I, J, K) be a hyperkähler manifold, and $L = aI + bJ + cK$ a generic induced complex structure. Then any stable bundle on (M, L) is hyperholomorphic.

Twistor transform and hyperholomorphic bundles 1: direct twistor transform

CLAIM: Let $\sigma : \text{Tw}(M) \rightarrow M$ be the standard projection, where M is hyperkähler or quaternionic-Kähler, and $\eta \in \Lambda^2 M$ a 2-form. Then $\sigma^*\eta$ is a **(1,1)-form iff η is $SU(2)$ -invariant.**

COROLLARY: Let (B, ∇) be a bundle with connection, and $\sigma^*B, \sigma^*\nabla$ its pullback to $\text{Tw}(M)$. **Then $(\sigma^*B, \sigma^*\nabla)$ has (1,1)-curvature iff ∇ has $SU(2)$ -invariant curvature.**

REMARK: This construction produces a holomorphic vector bundle on $\text{Tw}(M)$ starting from a connection with $SU(2)$ -invariant curvature. It is called **direct twistor transform**. The **inverse twistor transform** produces a bundle with connection on M from a holomorphic bundle on $\text{Tw}(M)$.

DEFINITION: **A non-Hermitian hyperholomorphic connection** on a complex vector bundle B is a connection (not necessarily Hermitian) which has $SU(2)$ -invariant curvature.

Twistor transform and hyperholomorphic bundles 2: inverse twistor transform

DEFINITION: Let M be a hyperkähler or quaternionic-Kähler manifold, and $\sigma : \text{Tw}(M) \rightarrow M$ its twistor space. For each point $x \in M$, $\sigma^{-1}(x)$ is a holomorphic rational curve in $\text{Tw}(M)$. It is called **a horizontal twistor line**.

THEOREM: (The inverse twistor transform; Kaledin-V.) Let B be a holomorphic vector bundle on $\text{Tw}(M)$, which is trivial on any horizontal twistor line. Denote by B_0 the C^∞ -bundle on M with fiber $H^0(B|_{\sigma^{-1}(x)})$ at $x \in M$. **Then B_0 admits a unique non-Hermitian hyperholomorphic connection ∇** such that B is isomorphic (as a holomorphic vector bundle) to its twistor transform $(\sigma^*B_0, (\sigma^*\nabla)^{0,1})$.

REMARK: The condition of being trivial on any horizontal twistor line is **open**. Therefore, **a holomorphic bundle on a $\text{Tw}(M)$ is “more or less the same” as a bundle with non-Hermitian hyperholomorphic connection on M .**

QUESTION: What can be said about the geometry of the moduli of holomorphic bundles on $\text{Tw}(M)$?

Rational curves on twistor spaces

From now on, we always assume that M is **hyperkähler** (and not quaternionic-Kähler).

DEFINITION: Denote by $\text{Sec}(M)$ **the space of holomorphic sections** of the twistor fibration $\text{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$. For each point $m \in M$, one has a **horizontal section** $C_m := \{m\} \times \mathbb{C}P^1$ of π . The space of horizontal sections is denoted $\text{Sec}_{hor}(M) \subset \text{Sec}(M)$

REMARK: The space of horizontal sections of π is identified with M . The normal bundle $NC_m = \mathcal{O}(1)^{\dim M}$. Therefore, **some neighbourhood of $\text{Sec}_{hor}(M) \subset \text{Sec}(M)$ is a smooth manifold of dimension $2 \dim M$.**

Let B be a (Hermitian) hyperholomorphic bundle on M , and W the deformation space of B , which is hyperkähler. Denote by \tilde{B} the holomorphic bundle on $\text{Tw}(M)$, obtained as a twistor transform of B . **Any deformation \tilde{B}_1 of \tilde{B} gives a holomorphic map $\mathbb{C}P^1 \rightarrow \text{Tw}(W)$ mapping $L \in \mathbb{C}P^1$ to a bundle $\tilde{B}_1|_{(M,L)} \subset \text{Tw}(M)$, considered as a point in (W, L) .**

THEOREM: (Kaledin-V.) This construction identifies **deformations of \tilde{B} (with appropriate stability conditions) and rational curves $S \in \text{Sec}(W)$.** The twistor transforms of Hermitian hyperholomorphic bundles on M correspond to $\text{Sec}_h(W) \subset \text{Sec}(W)$.

Holomorphic bundles on $\mathbb{C}P^3$ and twistor sections

DEFINITION: An instanton on $\mathbb{C}P^2$ is a stable bundle B with $c_1(B) = 0$. A framed instanton is an instanton equipped with a trivialization $B|_C$ for a line $C \subset \mathbb{C}P^2$.

THEOREM: (Nahm, Atiyah, Hitchin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{C}P^2$ is **smooth, connected, hyperkähler**.

The main result of today's talk can be stated as follows

METATHEOREM: There is a similar correspondence between the holomorphic bundles on $\text{Tw}(\mathbb{H}) = \mathbb{C}P^3 \setminus \mathbb{C}P^1$, with appropriate stability and framing conditions, and twistor sections in $\text{Sec}(\mathcal{M}_{r,c})$.

Mathematical instantons

DEFINITION: A **mathematical instanton bundle** on $\mathbb{C}P^n$ is a locally free coherent sheaf E on $\mathbb{C}P^n$ with $c_1(E) = 0$ satisfying the following cohomological conditions:

1. for $n \geq 2$, $H^0(E(-1)) = H^n(E(-n)) = 0$;
2. for $n \geq 3$, $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$;
3. for $n \geq 4$, $H^p(E(k)) = 0$, $2 \leq p \leq n-2$ and $\forall k$;

The integer $c = -\chi(E(-1)) = h^1(E(-1)) = c_2(E)$ is called **the charge** of E .

A framed instanton is a mathematical instanton equipped with a trivialization of $B|_\ell$ for some fixed line $\ell = \mathbb{C}P^1 \subset \mathbb{C}P^n$.

REMARK: Mathematical instantons of rank 2 **are always stable** (follows from the monad description below).

REMARK: The space $\mathbb{M}_{r,c}$ of framed instantons with charge c and rank r **is a principal $SL(2)$ -bundle** over the space of all mathematical instantons trivial on ℓ .

THEOREM: (Jardim–V.) The space \mathbb{M}_c of framed rank r mathematical instantons on $\mathbb{C}P^3$ **is naturally identified with the space of twistor sections $\text{Sec}(\mathcal{M}_{r,c})$.**

Monads and mathematical instantons

DEFINITION: A monad is a sequence of vector bundles $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ which is exact in the first and the last term. The cohomology of a monad is $\ker j / \operatorname{im} i$.

THEOREM: Let B be a holomorphic bundle of rank 2 on $\mathbb{C}P^n$, $c_1(B) = 0$, $c_2(B) = c$. Then the following conditions are equivalent.

(i) B is a mathematical instanton.

(ii) B is a cohomology of a monad

$$0 \longrightarrow V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}P^k}(-1) \longrightarrow W \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}P^k} \longrightarrow U \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{C}P^k}(1) \longrightarrow 0$$

with $\dim V = \dim U = c$ and $\dim W = 2c + 2$.

ADHM construction

DEFINITION: Let V and W be complex vector spaces, with dimensions c and r , respectively. The **ADHM data** is maps

$$A, B \in \text{End}(V), I \in \text{Hom}(W, V), J \in \text{Hom}(V, W).$$

We say that ADHM data is

stable,

if there is no subspace $S \subsetneq V$ such that $A(S), B(S) \subset S$ and $I(W) \subset S$;

costable,

if there is no nontrivial subspace $S \subset V$ such that $A(S), B(S) \subset S$ and $S \subset \ker J$;

regular,

if it is both stable and costable.

The ADHM equation is $[A, B] + IJ = 0$.

THEOREM: (Atiyah, Drinfeld, Hitchin, Manin) Framed rank r , charge c instantons on $\mathbb{C}P^2$ are in bijective correspondence with the set of equivalence classes of regular ADHM solutions. In other words, **the moduli of instantons on $\mathbb{C}P^2$ is identified with moduli of the corresponding quiver representation.**

The multi-dimensional ADHM construction

DEFINITION: Let V and W be complex vector spaces, with dimensions c and r , respectively. The **d -dimensional ADHM data** is maps

$$A_k, B_k \in \text{End}(V), I_k \in \text{Hom}(W, V), J_k \in \text{Hom}(V, W), (k = 0, \dots, d)$$

Choose homogeneous coordinates $[z_0 : \dots : z_d]$ on $\mathbb{C}P^d$ and define

$$\tilde{A} := A_0 \otimes z_0 + \dots + A_d \otimes z_d \quad \text{and} \quad \tilde{B} := B_0 \otimes z_0 + \dots + B_d \otimes z_d.$$

We say that d -dimensional ADHM data is

globally regular, if $(\tilde{A}_p, \tilde{B}_p, \tilde{I}_p, \tilde{J}_p)$ is regular for every $p \in \mathbb{C}P^d$. The **d -dimensional ADHM equation** is $[\tilde{A}_p, \tilde{B}_p] + \tilde{I}_p \tilde{J}_p = 0$, for all $p \in \mathbb{C}P^d$

THEOREM: (Marcos Jardim, Igor Frenkel) Let $C_d(r, c)$ denote the set of globally regular solutions of the d -dimensional ADHM equation. Then **there exists a 1-1 correspondence between equivalence classes of globally regular solutions of the d -dimensional ADHM equations and isomorphism classes of rank r instanton bundles** on $\mathbb{C}P^{d+2}$ framed at a fixed line ℓ , where $\dim W = \text{rk}(E)$ and $\dim V = c_2(E)$.

The multi-dimensional ADHM construction for $d = 1$

For $d = 1$, we obtain that the d -dimensional ADHM solutions are families of solutions of ADHM parametrized by $\mathbb{C}P^3$. Also, the space of 1-dimensional ADHM data is the space of sections of

$$\mathcal{O}(1) \otimes_{\mathbb{C}} \left[\text{Hom}(W, V) \oplus \text{Hom}(V, W) \oplus \text{End}(V) \oplus \text{End}(V) \right]$$

over $\mathbb{C}P^1$, that is, the twistor space of a quaternionic vector space $U = \text{Hom}(W, V) \oplus \text{Hom}(V, W) \oplus \text{End}(V) \oplus \text{End}(V)$. Now, the hyperkähler structure on 0-dimensional ADHM solutions for each $p \in \mathbb{C}P^1$ is compatible with the hyperkähler structure on U , because the space of 0-dimensional ADHM solutions is obtained from U by hyperkähler reduction. **This is used to prove the theorem about instantons on $\mathbb{C}P^3$ and twistor sections.**