Hyperbolic groups are not Ulam stable

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joint work with Michael Brandenbursky

Ulam stabulity

DEFINITION: (D. Kazhdan)
Let V be a Hermitian space. An **ε-representation** of a group Γ as a map $\rho: \Gamma \to U(V)$ satisfying $d(\rho(xy), \rho(x)\rho(y)) < \varepsilon$, where $\|\cdot\|$ is a left-invariant metric on $U(V)$ associated with an operator norm.

DEFINITION: Define the distance between two maps $\rho_1, \rho_2: \Gamma \to U(V)$ as $d(\rho_1, \rho_2) := \sup_{x \in G} d(\rho_1(x), \rho_2(x))$. The group Γ is called **Ulam stable** if for any $\delta > 0$ there exists $\varepsilon > 0$ such that any finite-dimensional ε-representation $q: \Gamma \to U(V)$ can be δ-approximated by a representation $\rho: \Gamma \to U(V)$.

QUESTION: (V. Milman)
Which groups are Ulam stable?

Answer: (D. Kazhdan)
Amenable groups are Ulam stable.

Answer: (D. Kazhdan)
Let $\Gamma = \pi_1(S)$, where S is a compact Riemann surface of genus 2. Then for each $\varepsilon > 0$ **there exists an ε-representation of Γ, which cannot be $\frac{1}{10}$-approximated.**

Hyperbolic groups are not Ulam stable

The main result for today:

THEOREM: (Brandenbursky - V.)
Let M be a compact manifold of negative sectional curvature, G a positive-dimensional connected Lie group. Put a left-invariant metric on G such that the diameter of any compact one-parametric subgroup is 1. **Then for each $\varepsilon > 0$, there exists an ε-representation of $\pi_1(M)$ which cannot be $1/3$-approximated by a representation.**

DEFINITION: (Gromov)
A quasimorphism is a map $q: G \to \mathbb{R}$ which satisfies $|q(xy) - q(x) - q(y)| < C$, where C is a constant independent from x, y.

We generalize this notion to

DEFINITION: An **(Ulam) quasimorphism** is a map q from a group Γ to a topological group G such that $q(xy)q(y)^{-1}q(x)^{-1}$ belongs to a fixed compact subset of G.

To solve Milman's problem, **we construct a new class of Ulam quasimorphisms, associated with vector bundles on manifolds of strictly negative sectional curvature.**

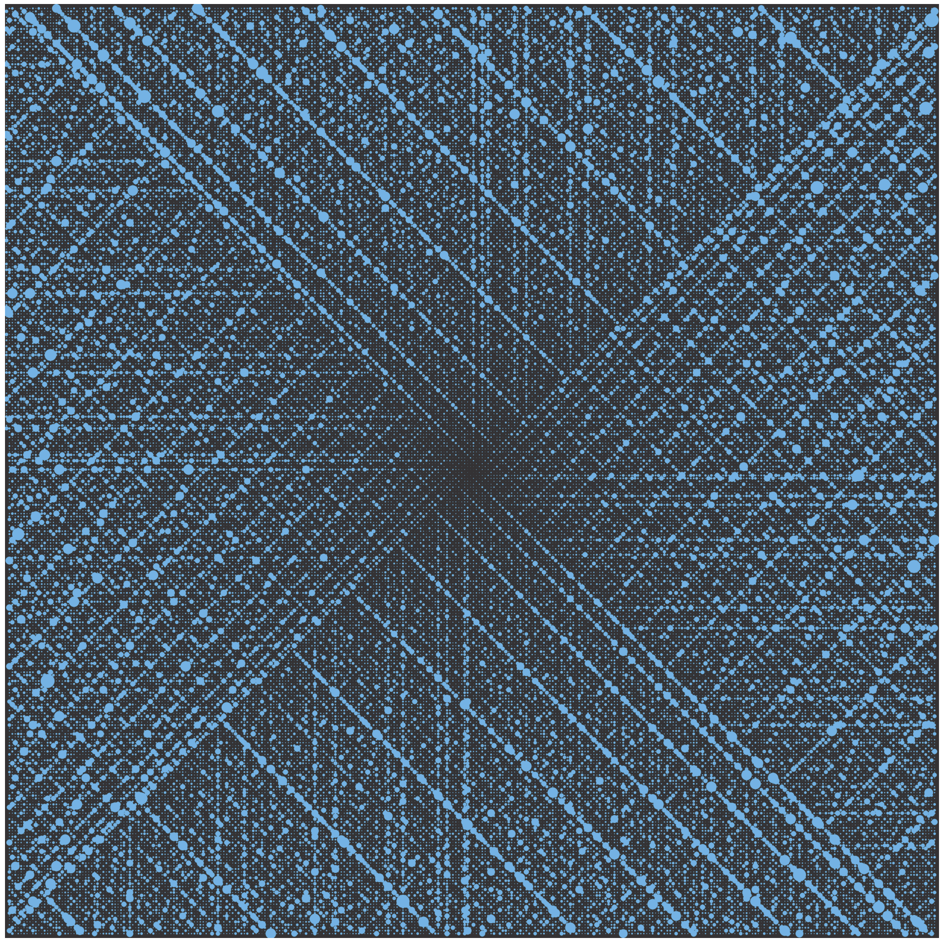
Stanisław Ulam (1909-1984)

Stanisław and Françoise Ulam, 1940s

Stanisław Ulam (1909-1984)

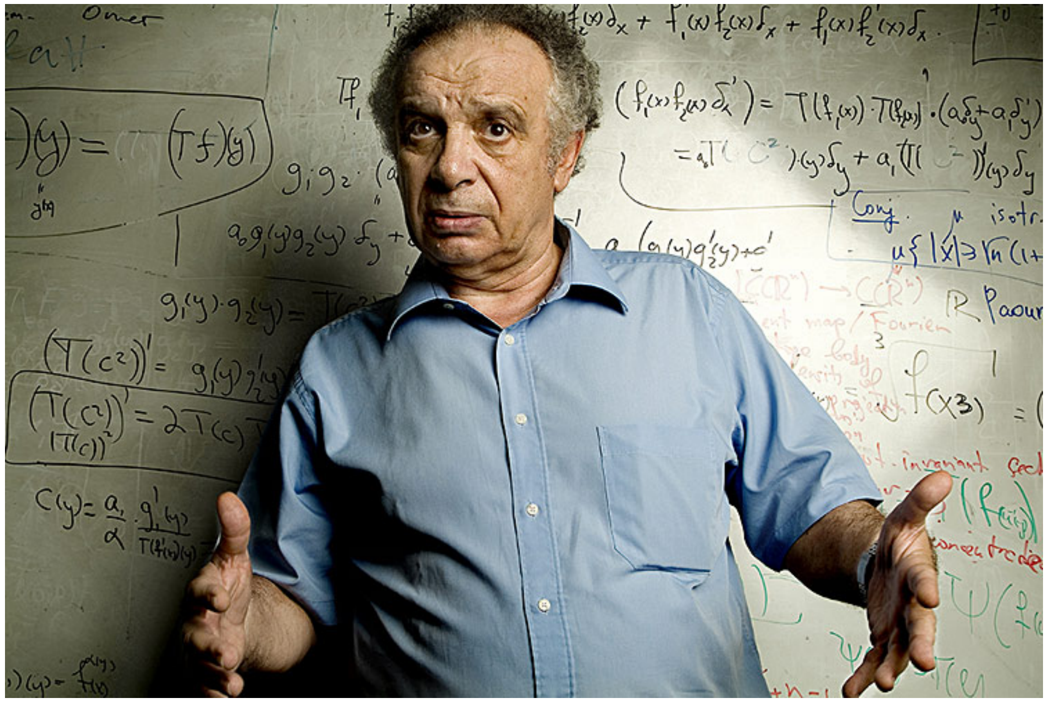
Stanisław Ulam and the hydrogen bomb

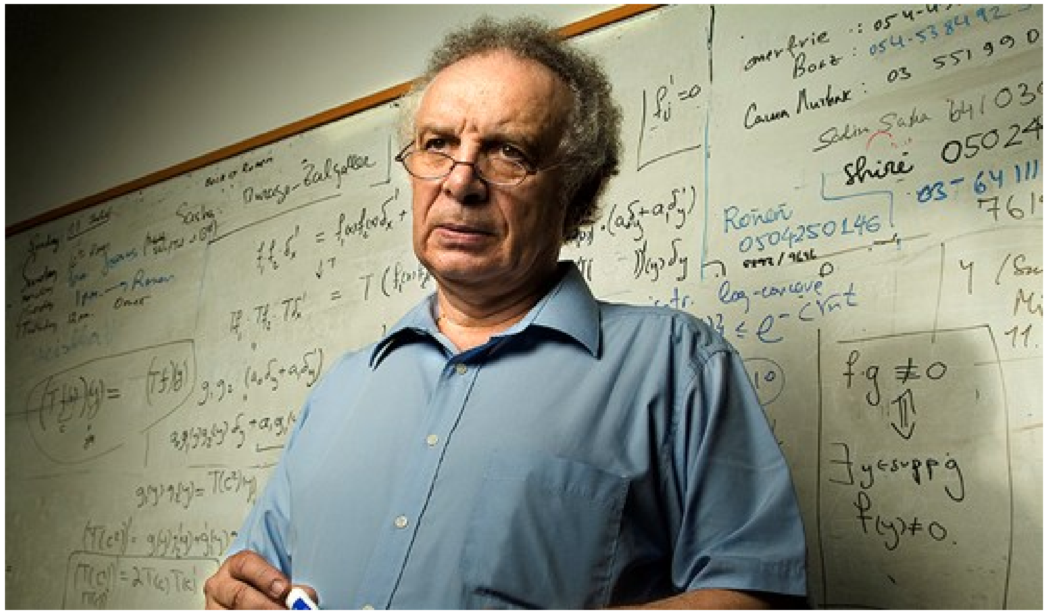
Stanisław Ulam (1909-1984)



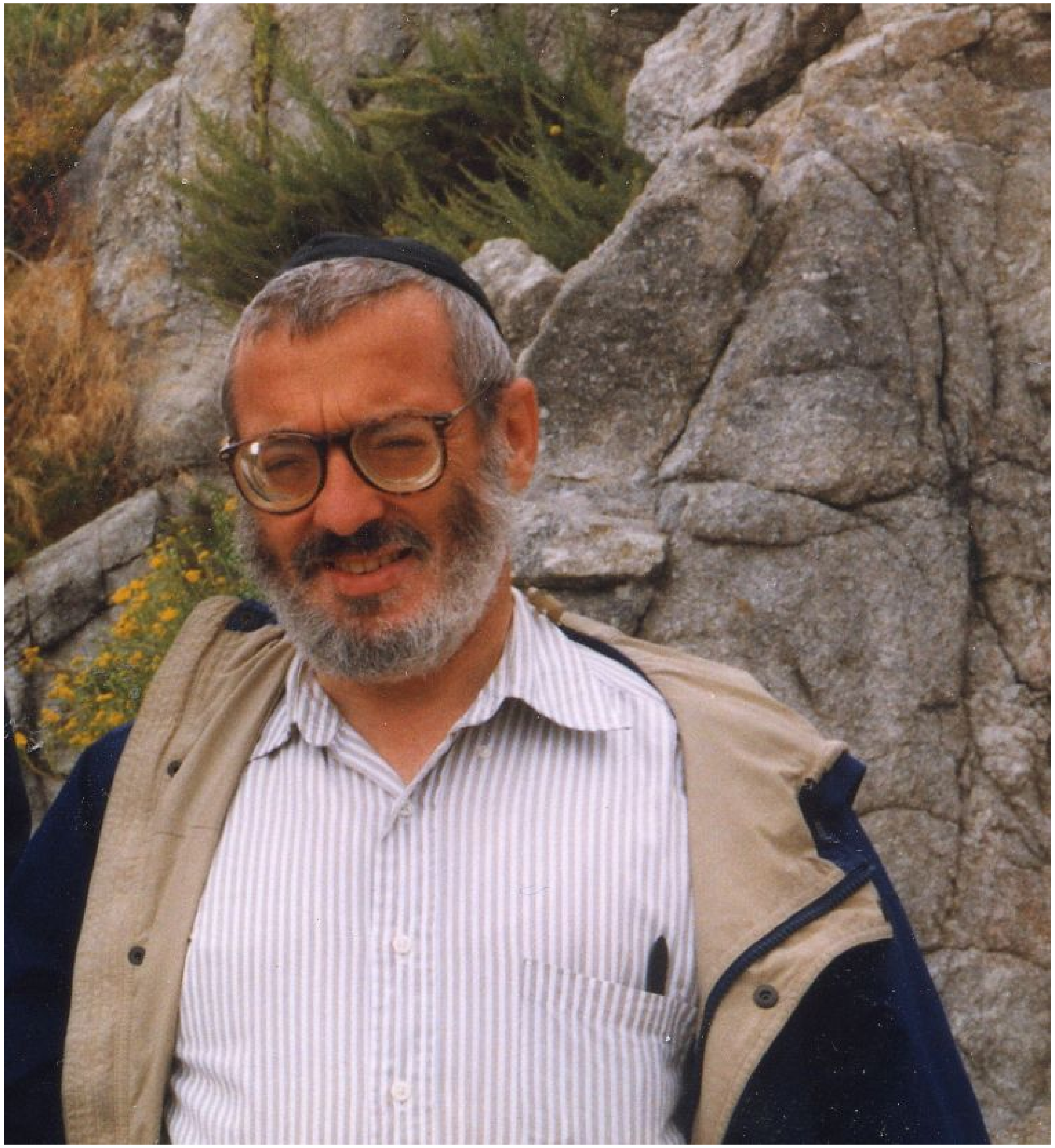
…The Ulam spiral or prime spiral is a graphical depiction of the set of prime numbers, devised by mathematician Stanisław Ulam in 1963 and popularized in Martin Gardner's Mathematical Games column in Scientific American a short time later. It is constructed by writing the positive integers in a square spiral and specially marking the prime numbers.

Vitaly Milman (b. 1939)





David Kazhdan (b. 1946)



Cartan-Hadamard theorem

THEOREM: (Cartan-Hadamard)
Let M be a complete, simply connected Riemannian manifold of non-positive sectional curvature. **Then M is contractible.**

Proof. Step 1: Let $\gamma_1 : [a,b]\to M$ and $\gamma_2 : [c,d]\to M$ be segments of geodesics in M, parametrized by the arc length. Gromov's CAT inequalities imply that the distance function $D : [a,b] \times [c,d]\to \mathbb{R}^{>0}$ taking x, y to $d(\gamma_1(x), \gamma_2(y))$ **is strictly convex,** unless the geodesics γ_1, γ_2 are segments of the same geodesic line.

Step 2: This implies that **any two points are connected by a unique geodesic:** indeed, if γ_1 and γ_2 have the same ends, $\gamma_1(a) = \gamma_2(c)$ and $\gamma_1(b) = \gamma_2(d)$ the function D would be equal to 0 in (a,c) and (b,d), hence it is zero on the diagonal, and the images of γ_1 and γ_2 coincide.

Step 3: Fix a reference point $p \in M$ and consider the function $H : M \times [0,1]\to M$ taking $x \in M$ and $t \in [0,1]$ to $\gamma(t\cdot d(p,x))$, where $\gamma : [0, d(p,x)]$ is the geodesic connecting p to x. **Convexity of D implies that H is continuous;** clearly, H is a deformation retract of M to p, hence M is contractible. ■

REMARK: Further on, we tacitly assume that **the base manifold of strictly negative curvature has dimension at least 2.**

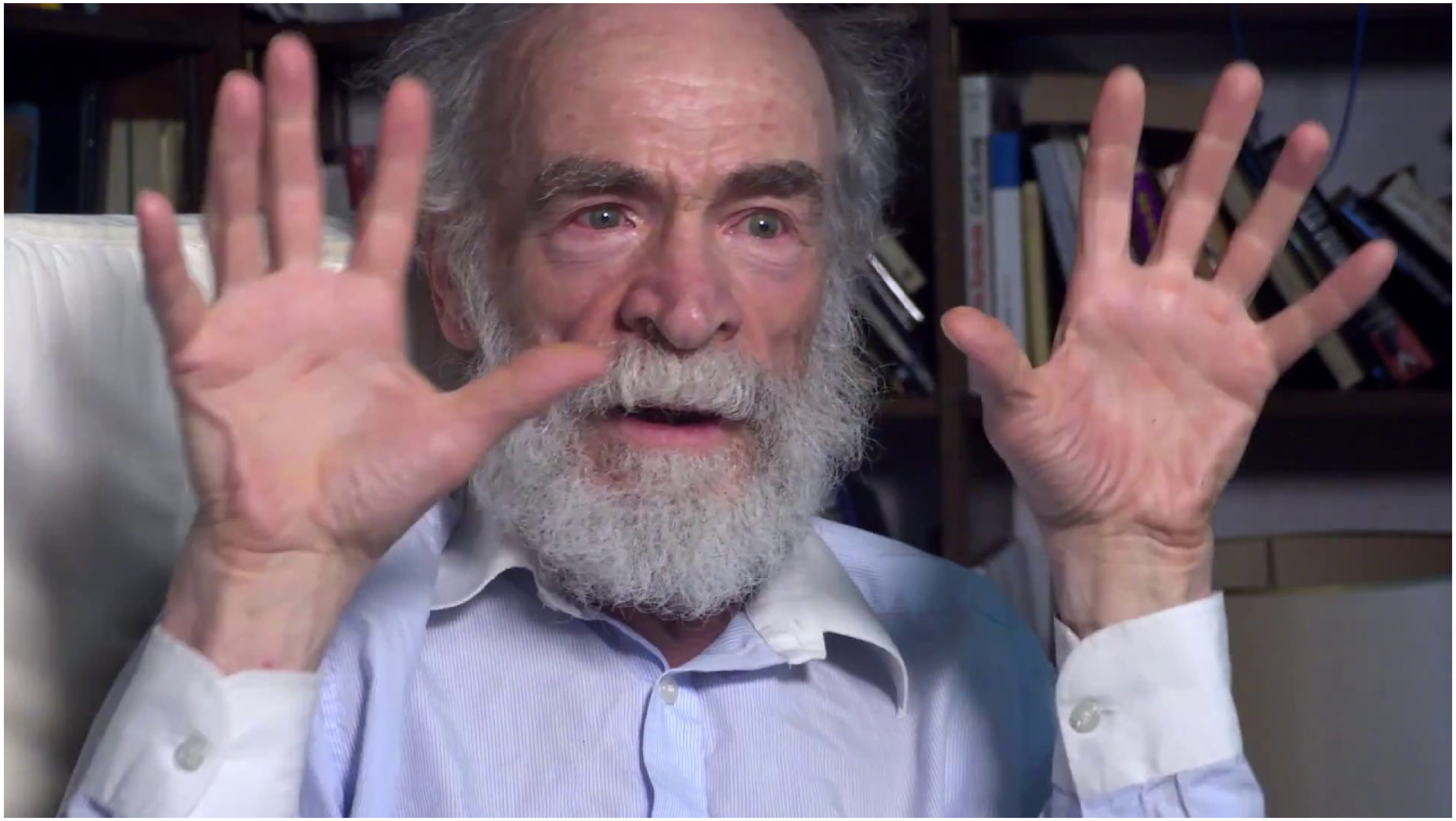
Area of a geodesic triangle

DEFINITION: Let M be a simply connected, complete manifold of non-positive sectional curvature. Let γ be the geodesic segment connecting b to c. The **geodesic triangle** $\Delta(a,b,c)$, associated with the points $a,b,c \in M$ is the union $\cup_{t\in[0,1]}[H_a(t)(b), H_a(t)(c)]$, where $H_a(t):\ M \to M$ is the homotopy along geodesics passing through a, and $[H_a(t)(b), H_a(t)(c)]$ the geodesic segment connecting $H_a(t)(b)$ and $H_a(t)(c)$.

THEOREM: (Gromov)
Let M be a simply connected, complete manifold of strictly negative sectional curvature $K(M) < -\varepsilon < 0$, and $\Delta(a,b,c)$ a geodesic triangle defined above. **Then the Riemannian area** $\operatorname{Area}(\Delta(a,b,c))$ **satisfies** $\operatorname{Area}(\Delta(a,b,c)) \leqslant \pi\varepsilon^{-2}$.

Proof: Uses the convexity of the map $D:\ [a,b]\times[c,d] \to \mathbb{R}^{>0}$, implied by CAT inequalities. ■

Mikhail Leonidovich Gromov (b. 1943)



Holonomy in a geodesic polygon

THEOREM: Let M be a compact manifold of negative sectional curvature, and Θ a geodesic n-polygon in M, that is, a contractible loop of n geodesic segments. Consider a principal G-bundle (P, ∇) with connection on M, and let $h(\Theta) \in G$ be the holonomy along the boundary of Θ, considered as a loop starting and ending at $p \in \Theta$. **Then $h(\Theta)$ belongs to a compact $K_n \subset G$ which is independent from the choice of Θ, but depends on the choice of n and (P, ∇) and the bound on the curvature of M.**

Proof. Step 1: By the effective version of the Ambrose-Singer theorem, the holonomy along a path **is bounded by the integral of the curvature over a disk filling this path.**

Step 2: The area of any geodesic triangle is bounded by Gromov's theorem. The absolute value of the curvature of ∇ is bounded from above because M is compact. **This implies that $h(\Theta)$ belongs to a fixed compact K when $n = 3$ and Θ is a triangle.**

Step 3: When $n > 3$, we represent Θ as a boundary of the union of $n-2$ geodesic triangles $D_1, .., D_{n-2}$ with common vertex p. **Then the holonomy $h(\Theta)$ is obtained as a product $h(\Theta) = h(D_1)h(D_2)...h(D_{n-2}) \subset K^{n-2}$. Therefore, $h(\Theta)$ belongs to a fixed compact $K_n := K^{n-2}$, independent from the choice of Θ.** ■

Non-commutative Barge-Ghys quasimorphism

DEFINITION: Let M be a compact manifold with non-positive sectional curvature, and (P, ∇) a principal G-bundle with connection. Fix $x \in M$. The **non-commutative Barge-Ghys map** takes $\gamma \in \pi_1(M, x)$ to the holonomy of ∇ along the geodesic path starting and ending at x and homotopic to γ.

REMARK: Since M has non-positive curvature, **this geodesic path is unique in its homotopy class** (Cartan-Hadamard).

Non-commutative Barge-Ghys quasimorphism (2)

COROLLARY: Let M be a compact manifold of negative sectional curvature, and (P,∇) a principal G-bundle with connection. Fix $x\in M$, and let $q:\ \pi_1(M){\to}G$ be the non-commutative Barge-Ghys map associated with (P,∇). **Then q is an Ulam quasimorphism.**

Proof. Step 1: Denote by $\tilde M \xrightarrow{\pi} M$ the universal cover of M. Let $a,b\in\pi_1(M)$, and P_a, $P_b\in$ Diff$(\tilde M)$ the corresponding deck transformations. Fix a preimage $\tilde x\in\pi^{-1}(x)$, and denote by $(\tilde P,\tilde\nabla)$ the pullback of (P,∇) to $\tilde M$. **By definition, the product $q(ab)q(b)^{-1}q(a)^{-1}$ is represented by the holonomy of $(\tilde P,\tilde\nabla)$ along the geodesic triangle connecting the points $\tilde x, P_a(\tilde x)$, and $P_b(P_a(\tilde x))$ in $\tilde M$.**

Step 2: By the previous theorem, **this quantity belongs to a compact subset independent from the choice of $x\in M$ and $a,b\in\pi_1(M)$.** ■

REMARK: To prove Ulam non-stability, **we need to be able to compute q explicitly**. We will modify this definition to arrive at a map which is easier to determine.

DEFINITION: A quasimorphism $q:\ \Gamma{\to}G$ is called **homogeneous** if $q(x^n)=q(x)^n$ for all $n\in\mathbb{Z}$.

Homogeneous Barge-Ghys quasimorphisms

DEFINITION: An element $\gamma \in \Gamma$ is called **primitive** if it cannot be represented as a power $\gamma = \varphi^n$, for any $n > 1$.

REMARK: Let M be a manifold of strictly negative curvature. Then $\pi_1(M)$ has no torsion. Moreover, **every element $x \in \pi_1(M)$ is a power of a primitive element.**

Given a primitive $\gamma \in \pi_1(M)$, let F_γ be the shortest free geodesic loop representing γ. By standard results of Riemannian geometry, F_γ is unique in every conjugacy class. For each conjugacy class of γ we fix a choice of a point $x \in F_\gamma$.

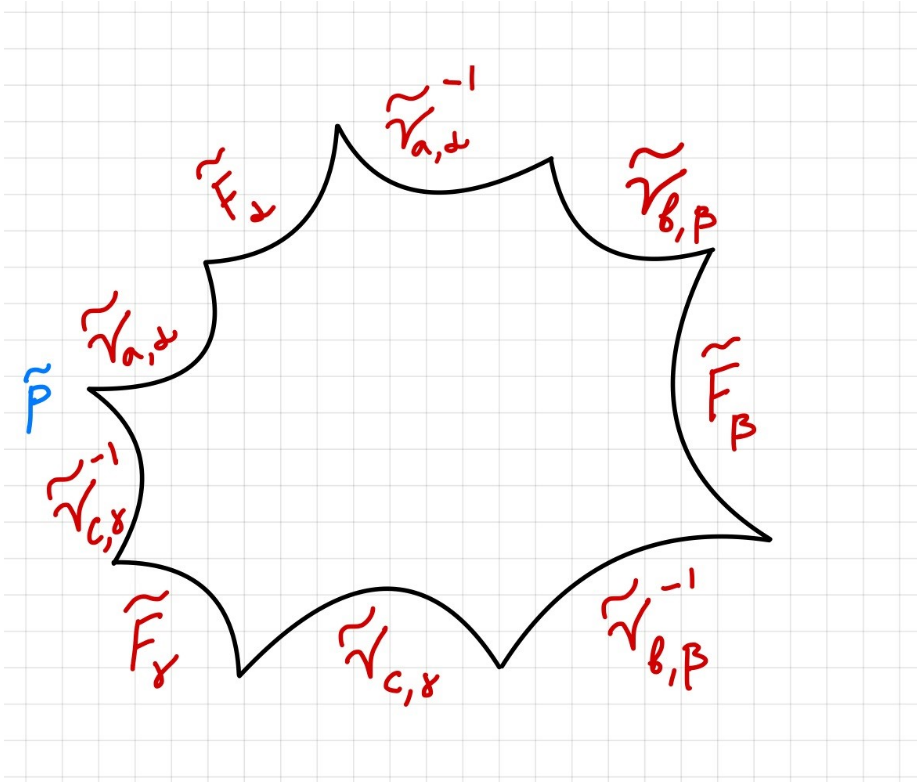
Fix a point $p \in M$, and $\gamma \in \pi_1(M)$. We are going to define a homogeneous quasimorphism $q : \Gamma \to GL(B_p)$, where B_p denotes the fiber of B in p, as follows. Let $\tilde{F}_\gamma := \nu_{x,\gamma} \circ F_\gamma \circ \nu_{x,\gamma}^{-1}$ be the 3-segment piecewise geodesic path obtained by connecting p to x, going around the loop F_γ starting and ending in x, and going back to p along $\nu_{x,\gamma}$ in the opposite direction. Clearly, this path represents γ in $\pi_1(M,p)$. Denote by $q(\gamma) \in GL(B_p)$ the holonomy along $\tilde{F}_\gamma$. By construction, q restricted to a cyclic subgroup is always a homomorphism.

DEFINITION: This map is called **the HBG-map associated with the connection ∇.**

Homogeneous Barge-Ghys quasimorphisms (2)

THEOREM: Let M be a compact manifold with strictly negative sectional curvature, $p \in M$ a base point, $\Gamma := \pi_1(M)$, and $q: \Gamma \to GL(B_p)$ the HBG map defined above. **Then $q: \Gamma \to GL(B_p)$ is a homogeneous Ulam quasimorphism.**

Proof. Step 1: Let $\alpha, \beta, \gamma = (\alpha\beta)^{-1}$ be elements of Γ. Choose any points $a \in F_\alpha$, $b \in F_\beta$, $c \in F_\gamma$. Then $q(\alpha)q(\beta)q(\alpha\beta)^{-1}$ is the holonomy of ∇ along a contractible geodesic polygon with 9 edges obtained by going along $\nu_{a,\alpha}, F_\alpha, \nu_{a,\alpha}^{-1}, \nu_{b,\beta}, F_\beta, \nu_{b,\beta}^{-1}, \nu_{c,\gamma}, F_\gamma, \nu_{c,\gamma}^{-1}$:



Homogeneous Barge-Ghys quasimorphisms (3)

Step 2: The holonomy along any contractible geodesic 9-gon **is bounded by a constant depending on the curvature of ∇ and M, hence q is an Ulam quasimorphism.**

Step 3: Homogeneity of q is clear, because $q(\gamma^n)$ is the holonomy of ∇ along the loop $\nu_{x,\gamma} \circ F_\gamma^n \circ \nu_{x,\gamma}^{-1}$. $\blacksquare$

Connections with prescribed holonomy

LEMMA: Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra, and P a trivial G-bundle on an interval $[0,1]$. Fix an element $g \in G$. Denote by ∇_0 the trivial connection on P. **Then there exists a $\mathfrak{g}$-valued 1-form A with compact support, such that the holonomy $\mathcal{H}\mathrm{ol}(\nabla)$ of the connection $\nabla := \nabla_0 + A$ is equal to g.**

Proof: Write A as $a(t)dt$, where $a \in \mathfrak{g}$ and dt is the standard 1-form on $[0,1]$. Then $\mathcal{H}\mathrm{ol}(\nabla) = \int_0^1 a(t)dt$. By Newton-Leibnitz formula, $\int_0^1 (\gamma(t)^{-1})^* \dot\gamma dt = g$. Setting $a(t) := (\gamma(t)^{-1})^* \dot\gamma$, we obtain a connection form which satisfies $\mathcal{H}\mathrm{ol}(\nabla) = \int_0^1 (\gamma(t)^{-1})^* \dot\gamma dt = g$. ■

Constructing HBG-quasimorphisms

THEOREM: Let M be a compact manifold of strictly negative curvature, G a non-abelian connected Lie group, and $x_1, ..., x_n \in \Gamma := \pi_1(M)$ a collection of primitive elements generating cyclic subgroups U_i satisfying $\forall u, v \in \Gamma, \forall i \neq j$, one has $U_i^u \cap U_j^v = 1$. Fix a collection of elements $g_i \in G$, with $i = 1, ..., n$. **Then there exists a connection ∇ on a trivial principal bundle P such that the corresponding HBG-quasimorphism $q_\nabla : \Gamma \to G$ takes x_i to g_i, $i = 1, 2, ..., n$.**

Proof: We are going to choose a connection ∇ on P such that the monodromy of ∇ along $\gamma_{x_i} = \nu_{p,x_i} F_{x_i} \nu_{p,x_i}^{-1}$ is equal to g_i.

Fix an open set B_{x_i} containing a segment of F_{x_i} and not intersecting the rest of the loops. We can choose B_i in such a way that it does not intersect the geodesic segment connecting p to p_i. Denote by ∇_0 the trivial connection on P. Using the previous lemma, **we modify ∇_0 by adding a 1-form with support in each B_i in such a way that the monodromy of ∇ along $\gamma_i \cap B_i$ is equal to g_i.** ■

Using HBG-quasimorphisms to prove that $\pi_1(M)$ is not Ulam stable

THEOREM: Let M be a compact manifold of negative sectional curvature, G a positive-dimensional connected Lie group. Choose a left-invariant metric on G such that the diameter of any closed subgroup is at least 1/3. Then for each $\varepsilon > 0$, there exists a connection ∇ such that **the corresponding HBG quasimorphism q_∇ is an ε-representation which cannot be 1/3-approximated by a representation.**

Proof. Step 1: Let $a_1, ..., a_n$ be the generators of $\pi_1(M)$. Find $b \in \pi_1(M)$ such that the cyclic group generated by b does not intersect the cyclic groups generated by a_i. Construct a connection ∇ on a trivial bundle such that the corresponding HBG-quasimorphism satisfies $q(a_i) = 0$, $q(b) = g$, where g is not a torsion element. Rescaling the connection form by $\frac{1}{m}$, **we can assume that the holonomy of ∇ is bounded by any given constant, and $q(b)^m = g$.** For m sufficiently large, this would give an ε-representation q_∇ such that $q_\nabla(a_i) = e$, and $q_\nabla(b)^m = g$.

Using HBG-quasimorphisms to prove that $\pi_1(M)$ is not Ulam stable (2)

Step 2: It remains to show that the ε-representation q_∇ cannot be $1/3$-aproximated by a representation ρ. By contradiction, assume that q_∇ is 1/3-aproximated by a representation $\rho: \pi_1(M) \to G$. Since $d(\rho(a_i^n), q_\nabla(a_i)^n) < 1/3$, and $q_\nabla(a_i) = 1$, the closure of a subgroup of G generated by $\rho(a_i)$ has diameter less than 1/3. Since the diameter of non-trivial subgroups of G is $\geqslant$ 1/3, this implies that $\rho(a_i) = \mathrm{Id}$. Therefore, $\rho(b) = \mathrm{Id}$. However, for all $n \in \mathbb{Z}$, we have $d(\rho(b)^n, q_\nabla(b)^n) < 1/3$, because q_∇ is 1/3-approximated by ρ. Then the diameter of the subgroup of G generated by $g = q_\nabla(b)$ is less than 1/3, which again implies that $g = \mathrm{Id}$, leading to contradiction. ■