Kähler manifolds,

lecture 1

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Connection and torsion

Notation: Let M be a smooth manifold, TM its tangent bundle, $\Lambda^i M$ the bundle of differential i-forms, $C^\infty M$ the smooth functions. **The space of sections of a bundle B is denoted by B.**

DEFINITION: A **connection** on a vector bundle B is a map $B \xrightarrow{\nabla} \Lambda^1 M \otimes B$ which satisfies

$$\nabla(fb) = df \otimes b + f\nabla b$$

for all $b \in B$, $f \in C^\infty M$.

REMARK: For any tensor bundle $\mathcal{B}_1 := B^* \otimes B^* \otimes ... \otimes B^* \otimes B \otimes B \otimes ... \otimes B$ **a connection on B defines a connection on $\mathcal{B}_1$** using the Leibniz formula:

$$\nabla(b_1 \otimes b_2) = \nabla(b_1) \otimes b_2 + b_1 \otimes \nabla(b_2).$$

DEFINITION: A torsion of a connection $\Lambda^1 \xrightarrow{\nabla} \Lambda^1 M \otimes \Lambda^1 M$ is a map $\mathrm{Alt} \circ \nabla - d$, where $\mathrm{Alt} : \Lambda^1 M \otimes \Lambda^1 M \longrightarrow \Lambda^2 M$ is exterior multiplication. It is a map $T_\nabla : \Lambda^1 M \longrightarrow \Lambda^2 M$.

An exercise: Prove that torsion is a $C^\infty M$-linear.

Linearized torsion map

DEFINITION: A torsor over a group G is a space X with a free, transitive action of G.

EXAMPLE: An affine space is a torsor over a linear space.

REMARK: If ∇_1 and ∇_2 are connections B, the difference $\nabla_-\nabla_2$ is $C^\infty M$-linear. **This makes the space $\mathcal{A}(B)$ of connections on B into an affine space,** that is, a torsor over a linear space $\Lambda^1(M) \otimes \operatorname{End}(B)$.

REMARK: Torsion is an affine map

$$\mathcal{A}(\Lambda^1 M) \longrightarrow \operatorname{Hom}(\Lambda^1 M, \Lambda^2 M) = TM \otimes \Lambda^2 M.$$

DEFINITION: An linearized torsion map is a map

$$T_{\nabla, lin}: \Lambda^1(M) \otimes \Lambda^1(M) \otimes TM \longrightarrow TM \otimes \Lambda^2 M$$

obtained as a linearization of a torsion map $\mathcal{A}(\Lambda^1 M) \longrightarrow \operatorname{Hom}(\Lambda^1 M, \Lambda^2 M)$.

REMARK: It is equal to

$$\operatorname{Alt} \otimes \operatorname{Id}_{TM}: \Lambda^1(M) \otimes \Lambda^1(M) \otimes TM \longrightarrow \Lambda^2 M \otimes TM.$$

Levi-Civita connection

DEFINITION: Let (M, g) be a Riemannian manifold. A connection ∇ is called **orthogonal** if $\nabla(g) = 0$. It is called **Levi-Civita** if it is torsion-free.

CLAIM: Orthogonal connection always exists, on any vector bundle B.

Proof: Take a covering $\{U_i\}$ such that $B\Big|_{U_i}$ are trivial and admit an orthonormal frame. Choose connections ∇_i locally on U_i fixing these frames. Then patch the local pieces together, using a splitting ψ_i of unit:

$$\nabla(b) = \sum \nabla_i(\psi_i b)$$

Exercise: Show that this defines a connection.

THEOREM: ("the main theorem of differential geometry")
For any Riemannian manifold, the Levi-Civita connection exists, and it is unique.

Levi-Civita connection (proof of existence and uniqueness)

Proof: Choose any orthogonal connection ∇_0. The space of all orthogonal connections is affine space modeled on $\Lambda^1 M \otimes \mathfrak{so}(TM)$.

Step 1: Identifying TM and $\Lambda^1 M$, obtain $\mathfrak{so}(TM) = \Lambda^2 M$.

Step 2: The linearized torsion map is

$$\mathrm{Alt} \otimes \mathrm{Id}_{TM}: \ \Lambda^1(M) \otimes \Lambda^2 M \longrightarrow \Lambda^2 M \otimes TM.$$

This is an isomorphism (dimension count, representation theory). Denote it by Ψ.

Step 3: Take $\nabla := \nabla_0 - \Psi^{-1}(T_{\nabla_0})$. Then $T_\nabla = T_{\nabla_0} - \Psi(\Psi^{-1}(T_{\nabla_0})) = 0$, hence **$\nabla$ is torsion-free**. ■

Principal G-bundles

DEFINITION: A **principal G-fibration** over M is a fibration $X \longrightarrow M$ with free action of G, transitively acting on fibers.

EXAMPLE: **A bundle of all frames** in an n-dimensional vector bundle B is principal a $GL(n)$-fibration. **A bundle of orthonormal frames** in an orthogonal n-dimensional vector bundle B is a principal $O(n)$-fibration.

REMARK: If B is a principal G_0-bundle, and $G \supset G_0$ is a Lie group containing G_0, **we can pass to a principal G-bundle** by taking $B := B_0 \times_{G_0} G$, where **$B_0 \times_{G_0} G$ denotes a quotient of $B_0 \times G$ by the diagonal action of G_0.**

DEFINITION: **A reduction** of a principal G-fibration B to $G_0 \subset G$ is a sub-bundle $B_0 \subset B$ which is a principal B_0 fibration, such that $B_0 \times_{G_0} G = B$.

EXAMPLE: The bundle of orthonormal frames gives a reduction of a principal $GL(n)$-bundle of all frames to $O(n)$.

DEFINITION: **A connection** on a principal G-bundle B is **a choice of a splitting $TB = T_{vert}B \oplus T_{hor}B$**, where $T_{vert}B$ is a bundle of **vertical tangent vectors** (vectors tangent to fibers of B).

G-structures

DEFINITION: A **G-structure** on a manifold is a reduction of a principal $GL(n)$-bundle of all frames to its subgroup $G \subset GL(n)$.

EXAMPLE: Riemannian structure: reduction of $GL(n)$ to $O(n)$.

EXAMPLE: An almost complex structure. Let $I: TM \longrightarrow TM$ be an operator satisfying $I^2 = -\operatorname{Id}$. Such an operator is called **an almost complex structure** on M. A bundle of all frames in $T^{1,0}(M) = \{x \in TM \otimes \mathbb{C} \mid I(x) = \sqrt{-1}\, x\}$ is a principal $GL(n, \mathbb{C})$-bundle, **giving a reduction of a structure group to $GL(n, \mathbb{C}) \subset GL(2n)$.**

EXAMPLE: An almost complex Hermitian structure. Let g be a Riemannian metric on an almost complex manifold satisfying $g(Ix, Iy) = g(x, y)$. Such a metric is called **Hermitian**. **A bundle of orthonormal frames in $T^{1,0}(M)$ gives a reduction of a structure group to $U(n) \subset GL(2n)$.**

G-connections

DEFINITION: A holonomy $\mathcal{H}\mathrm{ol}(\nabla)$ of a connection ∇ is a group acting on frames by parallel transport via ∇ along all loops in M.

DEFINITION: Fix a G-structure on an n-manifold M, with $G \subset GL(n)$. A connection ∇ on M is called **a G-connection**, or **compatible with a G-structure** if its holonomy preserves the bundle of frames compatible with G.

REMARK: When a G-structure is given by a set of tensors $\{\xi_i\}$ (such as in the case of almost complex, orthogonal, Hermitian), **this is equivalent to $\mathcal{H}\mathrm{ol}(\nabla)$ preserving these tensors, that is, $\nabla(\xi_i) = 0$.**

REMARK: Let $\mathfrak{g} \subset \mathrm{End}(TM)$ **be the Lie algebra of infinitesimal transforms of a G-structure W,** and ∇_1, ∇_2 be G-connections. Then $\nabla_1 - \nabla_2 \in \Lambda^1(M) \otimes \mathfrak{g}$. Converse is also true: **adding a $\mathfrak{g}$-valued 1-form to a G connection, we obtain another G-connection.**

CLAIM: The space $\mathcal{A}(W)$ of all G-connections is an affine space modeled on $\Lambda^1(M) \otimes \mathfrak{g}$.

Exercise: Prove that a G-connection always exists.

Intrinsic torsion

Let M be a manifold with a G-structure W, $\mathcal{A}(W)$ the space of all G-connections, $\Lambda^1(M) \otimes \mathfrak{g}$ its linearization, $\mathfrak{g} \subset \operatorname{End}(TM)$ the Lie algebra of infinitesimal transforms of W, and

$$T_{\nabla, lin}: \Lambda^1(M) \otimes \mathfrak{g} \longrightarrow \Lambda^2 M \otimes TM$$

the linearized torsion map.

DEFINITION: (Élie Cartan) **the space of intrinsic torsion** $\mathfrak{T}$ is a quotient of $\Lambda^2 M \otimes TM$ by the image of $\Lambda^1(M) \otimes \mathfrak{g}$, and **an intrinsic torsion** of a G-structure is an image of Tor_∇ in $\mathfrak{T}$, for some G-connection.

REMARK: **An intrinsic torsion of a G-structure vanishes if and only if there exists a torsion-free G-connection.**

CLAIM: (Élie Cartan) **An intrinsic torsion is the only first order invariant of a G-structure.** It vanishes if and only if a G-structure is flat up to second order terms.

Intrinsic torsion (examples)

EXAMPLE: For $G = O(n)$, the space of intrinsic torsion vanishes, because

$$T_{\nabla,lin}: \Lambda^1(M) \otimes \mathfrak{o}(n) \longrightarrow \Lambda^2 M \otimes TM$$

is an isomorphism.

EXAMPLE: When $G = Sp(2n, \mathbb{R})$, one has $\mathfrak{g} \cong \mathrm{Sym}^2(\Lambda^1(M))$, and **the intrinsic torsion space is a cokernel of a map**

$$T_{\nabla,lin}: \Lambda^1(M) \otimes \mathrm{Sym}^2(\Lambda^1(M)) \longrightarrow \Lambda^2 M \otimes \Lambda^1(M)$$

isomorphic to $\Lambda^3(M)$.

Exercise: Let G be $Sp(2n, \mathbb{R})$-structure defined by a non-degenerate skew-symmetric form ω. **Show that the corresponding intrinsic torsion term is $d\omega \in \Lambda^3(M)$.**

COROLLARY: An $Sp(2n, \mathbb{R})$-structure has no intrinsic torsion if and only if it is symplectic.

Complex manifolds

DEFINITION: Let M be a smooth manifold. An **almost complex structure** is an operator $I: TM \longrightarrow TM$ which satisfies $I^2 = -\operatorname{Id}_{TM}$.

The eigenvalues of this operator are $\pm\sqrt{-1}$. The corresponding eigenvalue decomposition is denoted $TM = T^{0,1}M \oplus T^{1,0}(M)$.

DEFINITION: An almost complex structure is **integrable** if $\forall X, Y \in T^{1,0}M$, one has $[X,Y] \in T^{1,0}M$. In this case I is called **a complex structure operator**. A manifold with an integrable almost complex structure is called **a complex manifold**.

THEOREM: (Newlander-Nirenberg)
This definition is equivalent to the usual one.

REMARK: The commutator defines a $\mathbb{C}^\infty M$-linear map $N := \Lambda^2(T^{1,0}) \longrightarrow T^{0,1}M$, called **the Nijenhuis tensor** of I. **One can represent N as a section of $\Lambda^{2,0}(M) \otimes T^{0,1}M$.**

Kähler manifolds

DEFINITION: An Riemannian metric g on an almost complex manifiold M is called **Hermitian** if $g(Ix, Iy) = g(x, y)$. In this case, $g(x, Iy) = g(Ix, I^2y) = -g(y, Ix)$, hence $\omega(x, y) := g(x, Iy)$ is skew-symmetric.

DEFINITION: The differential form $\omega \in \Lambda^{1,1}(M)$ is called **the Hermitian form** of (M, I, g).

THEOREM: Let (M, I, g) be an almost complex Hermitian manifold. **Then the following conditions are equivalent.**

(i) The complex structure I is integrable, and the Hermitian form ω is closed.

(ii) One has $\nabla(I) = 0$, where ∇ is the Levi-Civita connection

$$\nabla: \operatorname{End}(TM) \longrightarrow \operatorname{End}(TM) \otimes \Lambda^1(M).$$

DEFINITION: A complex Hermitian manifold M is called **Kähler** if either of these conditions hold. The cohomology class $[\omega] \in H^2(M)$ of a form ω is called **the Kähler class** of M.

Examples of Kähler manifolds.

Definition: Let $M = \mathbb{C}P^n$ be a complex projective space, and g a $U(n+1)$-invariant Riemannian form. It is called **Fubini-Study form on $\mathbb{C}P^n$**. The Fubini-Study form is obtained by taking arbitrary Riemannian form and averaging with $U(n+1)$.

Remark: For any $x \in \mathbb{C}P^n$, the stabilizer $St(x)$ is isomorphic to $U(n)$. Fubini-Study form on $T_x\mathbb{C}P^n = \mathbb{C}^n$ is $U(n)$-invariant, hence unique up to a constant.

Claim: Fubini-Study form is Kähler. Indeed, $d\omega|_x$ is a $U(n)$-invariant 3-form on $\mathbb{C}^n$, but such a form must vanish.

Corollary: Every projective manifold (complex submanifold of $\mathbb{C}P^n$) is Kähler.

REMARK: A manifold (M, I, g) is Kähler **if and only if the intrinsic torsion of its $U(n)$-structure vanishes.**

REMARK: A Kähler manifold is flat, up to second order terms: There is a map from $\mathbb{C}^n$ with a flat metric and a complex structure to M, compatible with g and I up to a second order. This is clear from Cartans' characterization of intrinsic torsion.

Intrinsic torsion of a Hermitian complex structure

For an almost complex Hermitian structure, $\mathfrak{g} = \mathfrak{u}(n)$ **is identified with** $\Lambda^{1,1}(M)$**.** This gives a linearized torsion map

$$T_{\nabla,lin}: \Lambda^1(M) \otimes \Lambda^{1,1}(M) \longrightarrow \Lambda^2 M \otimes \Lambda^1(M).$$

REMARK: The space of intrinsic torsion of an almost complex Hermitian structure is

$$\Lambda^{2,0} \otimes \Lambda^{0,1}(M) \oplus \Lambda^{0,2} \otimes \Lambda^{1,0}(M) \oplus \Lambda^{2,0} \otimes \Lambda^{1,0}(M) \oplus \Lambda^{0,2} \otimes \Lambda^{0,1}(M) = \\ = \Lambda^{2,1}(M) \oplus \Lambda^{1,2}(M) \oplus \Lambda^{2,0}(M) \otimes T^{0,1} \oplus \Lambda^{0,2}(M) \otimes T^{1,0}$$

CLAIM: Let (M, I, g) be a Hermitian structure, and $\mathfrak{T}$ its intrinsic torsion, decomposed as above, Then **the first two terms of $\mathfrak{T}$ are equal to $d\omega$,** where ω is a Hermitian form, and **the last two terms of $\mathfrak{T}$ are N and $\overline{N}$,** where N is a Nijenhuis tensor.

COROLLARY: Intrinsic torsion of a complex Hermitian structure (I, g) vanishes if and only if I is integrable and ω is closed.