Kähler manifolds,

lecture 2

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Group representations

DEFINITION: Let V be a finite-dimensional vector space, $G \xrightarrow{\rho} \operatorname{End}(V)$ a group representation. It is called **simple**, or **irreducible** if V has no non-trivial G-invariant subspaces $V_1 \subsetneq V$, and **semisimple** if V is a direct sum of simple representations.

EXAMPLE: If ρ is a unitary representation, it is semisimple. Indeed, an orthogonal complement $V_1{}^\perp$ of a G-invariant space is also G-invariant.

EXAMPLE: If G is a reductive Lie group, all its representations are semisimple. This is one of the definitions of a reductive group.

EXAMPLE: If G is a finite group, all its representations are semisimple. Take a positive definite scalar product ξ on V, and let

$$\xi_G(x, y) := \sum_{g \in G} \xi(gx, gy).$$

Then ξ_G is a G-invariant and positive definite, and V is semisimple as shown above.

The group algebra

DEFINITION: A group algebra of a finite group G is a vector space $\mathbb{C}[G]$ with basis $\{g_1, ...g_n\} = G$, and multiplication defined as in G. **We consider $\mathbb{C}[G]$ as a representation of G,** with G acting from the left.

THEOREM: Let V be an irreducible representation of G. **Then V is a sub-representation of $\mathbb{C}[G]$.**

Proof: Take $v \in V$, and let $\rho : \mathbb{C}[G] \longrightarrow V$ map $g \in \mathbb{C}[G]$ to $g(v)$. This map is non-zero, hence surjective. Taking duals, obtain an embedding $V \hookrightarrow \mathbb{C}[G]$. ■

COROLLARY: There is only finitely many pairwise non-isomorphic simple representations of a finite group G.

COROLLARY: The symmetric group S_2 has only 2 irreducible representations. Indeed, $\mathbb{C}[G]$ is 2-dimensional.

DEFINITION: A conjugacy class in a group is a set $\{g^{-1}hg \quad | \quad g \in G\}$, where h is fixed and g runs through all G.

THEOREM: (Frobenius) **The number of pairwise non-isomorphic simple representations of a finite group is equal to its number of conjugacy classes**.

Representations of a symmetric group

DEFINITION: An element $a \in A$ of an algebra is called **idempotent** if $a^2 = a$.

We fix a G-invariant positive definite Hermitian form on $\mathbb{C}[G]$.

REMARK: Let $V \subset \mathbb{C}[G]$ be a sub-representation, and $\iota : \mathbb{C}[G] \longrightarrow V$ be an orthogonal (hence, G-invariant) projection. Then $\iota^2 = \iota$, that is, ι is idempotent.

Representations of S_n are classified by idempotents in $\mathbb{C}[S_n]$ called **Young symmetrizers**.

DEFINITION: For any embedding $G \hookrightarrow S_n$, of symmetric group, consider an idempotent $a_G \in C[S_n]$ called **a symmetrizer**,

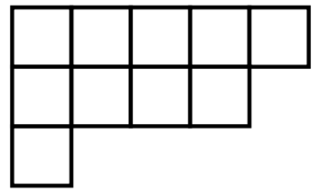
$$a_G(v) := \frac{1}{|G|} \sum_{g \in G} g(v)$$

and another one, called **antisymmetrizer**

$$b_G(v) := \frac{1}{|G|} \sum_{g \in G} sign(g) g(v)$$

Young symmetrizers

DEFINITION: Every partition of a set of n elements corresponds to a table called **a Young diagram**



Young diagram, corresponding to a partition of 10.

A **Young tableau** is obtained from a Young diagram by putting numbers from 1 to n arbitrarily in all squares.

1	2	4	7	8
3	5	6	9	
10				

Young tableau, corresponding to a partition of 10.

DEFINITION: Fix a Young tableau σ, and define the permutation groups P and $Q \subset S_n$ with P preserving each row and Q each column of a tableau. An element $c_\sigma := a_P b_Q$ is called **the Young symmetrizer**.

Specht modules

DEFINITION: An image of left multiplication by c_σ is a right S_n-subrepresentation $V_\sigma \subset \mathbb{C}[S_n]$, which is called **a Specht module**.

REMARK: The Young symmetrizer satisfies $c_\sigma^2 = \lambda c_\sigma^2$, where $\lambda = \frac{n!}{|P||Q| \dim V_\sigma}$.

THEOREM: Every irreducible representation of a symmetric group is isomorphic to a Specht module. Moreover, **equivalence classes of irreducible representations of a symmetric group are in 1 to 1 correspondence with Young diagrams.**

REMARK: This is essentially a special case of Frobenius theorem. Indeed, the conjugacy classes of permutations are classified by partitions of n.

Representations of $GL(n)$

THEOREM: Let V be a vector space, and $V^{\otimes n}$ its tensor power. **Then $V^{\otimes n}$ is a direct sum of irreducible $GL(V)$-representations, parametrized by Young diagrams.** Given a Young tableau σ, and the corresponding Young symmetrizer c_σ, the image $c_\sigma(V^{\otimes n})$ is irreducible.

EXAMPLE: $V \otimes V = \Lambda^2 V \oplus \operatorname{Sym}^2 V$.

EXAMPLE: $V \otimes V \otimes V = \Lambda^3 V \oplus \operatorname{Sym}^3 V \oplus K(V)$, where $K(V)$ **("the space of Cartan tensors")** can be defined as a kernel of $\operatorname{Sym}_{23}\Big|_{\Lambda^2 V \otimes V}$.

EXAMPLE: $V \otimes V \otimes V \otimes V = \Lambda^4 V \oplus \operatorname{Sym}^4 V \oplus V_{3,1} \oplus V_{2,1,1} \oplus V_{2,2}$. Here $V_{3,1} = \ker \operatorname{Sym}_{34}\Big|_{\Lambda^3 V \otimes V}$, $V_{3,1} = \ker \operatorname{Alt}_{34}\Big|_{\operatorname{Sym}^3 V \otimes V}$, and $V_{2,2}$ their orthogonal complement, **which can be defined as a space of tensors which are antysymmetric under interchange of 1 and 2 and 3 and 4, symmetrized under interchange of 1 and 4 and 3 and 4.**

CLAIM: This is the same as a kernel of a multiplication

$$\ker \left(\operatorname{Sym}^2(\Lambda^2 V) \longrightarrow \Lambda^4 V \right).$$

DEFINITION: The space $V_{2,2} \subset V \otimes V \otimes V \otimes V$ is called **the space of algebraic curvature tensors**.

Curvature of a connection

Let M be a manifold, B a bundle, $\Lambda^i M$ the differential forms, and $\nabla: B \longrightarrow B \otimes \Lambda^1 M$ a connection. We extend ∇ to $B \otimes \Lambda^i M \xrightarrow{\nabla} B \otimes \Lambda^{i+1} M$ in a natural way, using the formula

$$\nabla(b \otimes \eta) = \nabla(b) \wedge \eta + b \otimes d\eta,$$

and define **the curvature Θ_∇** of ∇ as $\nabla \circ \nabla: B \longrightarrow B \otimes \Lambda^2 M$.

CLAIM: This operator is $C^\infty M$-linear.

REMARK: We shall consider Θ_∇ as an element of $\Lambda^2 M \otimes \operatorname{End} B$, that is, an $\operatorname{End} B$-valued 2-form.

REMARK: Given vector fields $X, Y \in TM$, the curvature can be written in terms of a connection as follows

$$\Theta_\nabla(b) = \nabla_X \nabla_Y b - \nabla_Y \nabla_X B - \nabla_{[X,Y]} b.$$

CLAIM: Suppose that the structure group of B is reduced to its subgroup G, and let ∇ be a connection which preserves this reduction. This is the same as to say that the connection form takes values in $\Lambda^1 \otimes \mathfrak{g}(B)$. Then Θ_∇ lies in $\Lambda^2 M \otimes \mathfrak{g}(B)$.

Symmetries of Riemannian curvature

EXAMPLE: Let ∇ be a Levi-Civita connection of a Riemannian manifold. **Then Θ_∇ lies in $\Lambda^2 M \otimes \mathfrak{so}(TM) = \Lambda^2 M \otimes \Lambda^2 M$.**

PROPOSITION: (an algebraic Bianchi identity)
Let Θ_∇ be a curvature of a Levi-Civita connection. **Then**

$$Cyclic_{1,2,3}(\Theta_\nabla) := \Theta_\nabla(X,Y,Z,\cdot) + \Theta_\nabla(Y,Z,X,\cdot) + \Theta_\nabla(Z,X,Y,\cdot) = 0.$$

Proof: Choose X, Y, Z which commute. Then $\nabla_X Y = \nabla_Y X$, etc., because ∇ is torsion-free. Then

$$Cyclic_{1,2,3}(\Theta_\nabla(X,Y,Z))$$
$$= (\nabla_X\nabla_Y Z - \nabla_X\nabla_Z Y) + (\nabla_Y\nabla_Z X - \nabla_Y\nabla_X Y) + (\nabla_Z\nabla_X Y - \nabla_Z\nabla_Y X) = 0$$

■

REMARK: It was discovered by Ricci some years after Bianchi discovered the "differential Bianchi identity".

COROLLARY: **The Riemannian curvature tensor lies in the bundle $TM_{2,2}$ of algebraic curvature tensors.**

Proof: Its projection to all other components of $TM \otimes TM \otimes TM \otimes TM$ vanish.

■

Ricci decomposition

DEFINITION: Let V be a space with positive definite scalar product, $\dim V > 4$ and $V_{2,2}$ the space of algebraic curvature tensors. Consider the trace map $\operatorname{Tr}_{1,3}: V_{2,2} \longrightarrow Sym^2 V$ It is denoted by Ric and called **the Ricci curvature**. Its trace is called **the scalar curvature**.

PROPOSITION: Consider $V_{2,2}$ as a representation of $O(V)$. **Then $V_{2,2}$ has the following irreducible components:**

$$V_{2,2} = W \oplus Sym_0^2 V \oplus \mathbb{R},$$

where $W = \ker \operatorname{Ric}$, and an embedding $\operatorname{Sym}^2 V \longrightarrow V_{2,2}$ is Ric^*. ■

DEFINITION: Let M be a Riemannian manifold, and $\Theta = W + \operatorname{Ric}$ be the decomposition of its curvature tensor, with $\operatorname{Ric} = \operatorname{Ric}_0 + S$ a decomposition of Ric onto its traceless part and the scalar part. Then W is called **its Weyl curvature** (or **conformal curvature**), Ric **the Ricci curvature**, Ric_0 **the traceless Ricci curvature**, and S **the scalar curvature**.

REMARK: When $\dim V = 2$, $\dim V_{2,2} = 1$, when $\dim V = 3$, $W = 0$ and $V_{2,2} = Sym^2 V$. When $\dim V = 4$, one has a decomposition

$$V_{2,2} = W^+ \oplus W^- \oplus Sym_0^2 V \oplus \mathbb{R}.$$

because $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

Glossary of Riemannian geometry

Let (M, g) be a Riemannian manifold. **Riemannian manifolds are classified according to their curvature decomposition:** $\Theta = W + \mathrm{Ric}_0 + S$.

DEFINITION: If $W = 0$, a manifold is called **conformally flat**. **Such manifold is locally conformally equivalent to a flat manifold.**

DEFINITION: If $\mathrm{Ric}_0 = 0$, a manifold is called **Einstein**. For such a manifold, one has $\mathrm{Ric} = \lambda g$. **It is possible to show that $\lambda = const$.** The number λ is called **an Einstein constant**

DEFINITION: If $W = \mathrm{Ric}_0 = 0$, and $\Theta = S$, a manifold is called **a space form**, or **a manifold of constant sectional curvature**. It is locally isometric to a Riemannian sphere, to a hyperbolic space, or to $\mathbb{R}^n$.

DEFINITION: If $S = const$, a manifold has **constant scalar curvature**.

Ricci decomposition for Kähler manifolds

DEFINITION: Let (M, I, g) be a Hermitian almost complex manifold, that is, a Riemannian manifold with $I : TM \longrightarrow TM$, $I^2 = -\operatorname{Id}_{TM}$, where $g(Ix, Iy) = g(x, y)$. Recall that (M, I, g) is called **Kähler** if I is preserved by the Levi-Civita connection.

REMARK: The Kähler curvature is a section of a bundle $\ker \operatorname{Sym}^2(\Lambda^{1,1}M) \longrightarrow \Lambda^{2,2}M$.

DEFINITION: Let V be a complex Hermitian vector space, $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} C$ its complexification, and $\Lambda^* V_{\mathbb{C}} = \oplus \Lambda^{p,q} V_{\mathbb{C}}$ the Hodge decomposition on its Grassmann algebra. Define **the space of algebraic Kähler curvature tensors** K as the kernel of multiplicative map $\ker \operatorname{Sym}^2(\Lambda^{1,1}V) \longrightarrow \Lambda^{2,2}V_{\mathbb{C}}$.

CLAIM: K is irreducible as a $GL(V, \mathbb{C})$-representation.

REMARK: The Ricci curvature of a Kähler curvature tensor is a symmetric 2-tensor satisfying $\operatorname{Ric}(Iv, Iw) = \operatorname{Ric}(v, w)$. Denote the space of such tensors by Her. **Clearly, $g(\cdot, I\cdot) \in \Lambda^{1,1}(M)$, for any $g \in$ Her.** In other words, Her **is a space of pseudo-Hermitian forms.**

Ricci decomposition for Kähler manifolds (cont.)

CLAIM: Consider K as a representation of unitary group $U(V)$. Then K **is irreducibly decomposed in the following way:**

$$K = K_0 \oplus \mathrm{Her}_0 V \oplus \mathbb{C}\omega,$$

where Her denotes the space of traceless pseudo-Hermitian forms.

COROLLARY: Let $\Theta \in K$ be a Kähler curvature tensor. Consider the trace $\mathrm{Tr}_{X,Y} \Theta(X, IY, \cdot, \cdot) \in \Lambda^{1,1}(M)$. **Then** $\mathrm{Tr}_{X,Y} \Theta(X, IY, \cdot, \cdot) \in \Lambda^{1,1}(M)$ **is proportional to** $\mathrm{Ric}(\Theta) \in \mathrm{Her}$**,** if we identify Her with $\Lambda^{1,1}(M)$ using $g \longrightarrow g(\cdot, I\cdot)$ as above.

Proof: Follows from Schur's lemma. ■

Canonical bundle of a Kähler manifold

DEFINITION: Let (M, I) be a complex manifold, and L a holomorphic Hermitian bundle. The holomorphic structure is understood as an operator $\overline{\partial}: L \longrightarrow L \otimes \Lambda^{0,1}M$, and integrability condition is $\overline{\partial}^2: L \otimes \Lambda^{0,2}M$ satisfies $\overline{\partial}^2 = 0$. We call **a Chern connection** a Hermitian connection on L which satisfies $\nabla^{0,1} = \overline{\partial}$.

CLAIM: The Chern connection on any holomorphic Hermitian bundle exists and is unique.

REMARK: The curvature of a Chern connection is obviously an $\operatorname{End} L$-**valued (1,1)-form.**

DEFINITION: Let (M, I, g) be an Kaehler manifold, $\dim_{\mathbb{C}} M = n$. Consider the bundle $\Lambda^{n,0}M$ of holomorphic volume forms, It is called **the canonical bundle of M**.

CLAIM:

The Levi-Civita connection induces the Chern connection on $\Lambda^{n,0}M$.

Canonical bundle and its curvature

REMARK: Let B be a bundle with connection, $\Theta_B \in \Lambda^2 M \otimes \operatorname{End} B$ its curvature, $\det B := \Lambda^{\dim B} B$ its determinant bundle, equipped with induced connection, and $\Theta_{\det B} \in \Lambda^2 M$ the curvature on $\det B$. **Then $\Theta_{\det B} = \operatorname{Tr} \Theta_B$.** This is clear, because

$$\nabla(b_1 \wedge b_2 \wedge ... \wedge b_m) = \sum_i b_1 \wedge b_2 \wedge ... \wedge \nabla b_i \wedge ...$$

REMARK: From this remark it follows that the curvature $\Theta_{\Lambda^{n,0}M}$ is equal to $\operatorname{Tr}_{X,Y} \Theta(\cdot,\cdot,X,IY)$.

COROLLARY:
The curvature of a canonical bundle is proportional to Ric**.**