Kähler manifolds,

lecture 3

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Linear algebra on Hermitian manifolds

DEFINITION: Let (M, I) be an almost complex manifold, and $\Lambda^1(M) = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M)$ be the usual decomposition along the eigenvalues of I. **The Hodge decomposition** $\Lambda^k(M) = \bigoplus_{p+q-k} \Lambda^{p,q}(M)$ is defined by

$$\Lambda^{p,q}(M) = \Lambda^p(\Lambda^{1,0}(M)) \otimes \Lambda^q(\Lambda^{0,1}(M)).$$

DEFINITION: Let M be a Riemannian oriented manifold, dimension m. We consider $\Lambda^i(M)$ as orthogonal bundles with the metric induced from M. Let $\operatorname{Vol} \in \Lambda^m(M)$ be its Riemannian volume form. We define **the Hodge star operator** $*: \Lambda^i(M) \longrightarrow \Lambda^{m-i}(M)$ by $(\eta, \eta') \operatorname{Vol} = \eta \wedge *\eta'$.

DEFINITION: Let (M, I, g) be an almost complex Hermitian manifold, $\dim_{\mathbb{C}} M = n$, and $\omega \in \Lambda^{1,1}(M)$ its Hermitian form. Define **the Hodge operators** $L(\eta) := \omega \wedge \eta$, $\Lambda := *L*$ it Hermitian conjugate, and $H\big|_{\Lambda^k M} := (n-k)\operatorname{Id}$.

CLAIM: The Hodge operators satisfy the relations of the Lie algebra $\mathfrak{sl}(2)$:

$$[L, \Lambda] = H, \quad [H, L] = 2L, \quad [H, \Lambda] = -2\Lambda.$$

DEFINITION: The triple (L, Λ, H) is called **the Lefschetz $\mathfrak{sl}(2)$-triple.**

De Rham differential on Kaehler manifolds

DEFINITION: Let M be an almost complex manifold. From the Leibniz formula, we obtain that $d(\Lambda^{p,q}(M)) \subset \Lambda^{p+1,q}(M) \oplus \Lambda^{p,q+1}(M)$. **This gives the usual decomposition $d = \partial + \overline{\partial}$** (first component, second component).

DEFINITION: $d^c := -IdI$.

REMARK: One has $\partial + \overline{\partial} = d$, $\partial - \overline{\partial} = \sqrt{-1}\, d^c$.

THEOREM: The following statements are equivalent.

1. I is integrable. 2. $\partial^2 = 0$. 3. $\overline{\partial}^2 = 0$. 4. $dd^c = -d^c d$ 5. $dd^c = 2\sqrt{-1}\,\partial\overline{\partial}$.

DEFINITION: The operator dd^c is called **the pluri-Laplacian**.

THEOREM: Let M be a Kaehler manifold. One has the following identities **("Kodaira idenitities").**

$$[\Lambda, \partial] = \sqrt{-1}\,\overline{\partial}^*, \quad [L, \overline{\partial}] = -\sqrt{-1}\,\partial^*, \quad [\Lambda, \overline{\partial}^*] = -\sqrt{-1}\,\partial, \quad [L, \partial^*] = \sqrt{-1}\,\overline{\partial}.$$

Equivalently,

$$[\Lambda, d] = (d^c)^*, \qquad [L, d^*] = -d^c, \qquad [\Lambda, d^c] = -d^*, \qquad [L, (d^c)^*] = d.$$

Lie superalgebras and supercommutators

DEFINITION: An operator on a graded vector space is called **even** (**odd**) if it shifts the grading by even (odd) number. The **parity** $\tilde{a}$ of an operator a is 0 if it is even, 1 if it is odd.

DEFINITION: A **supercommutator** of operators on a graded vector space is defined by a formula $\{a,b\} = ab - (-1)^{\tilde{a}\tilde{b}} ba$.

DEFINITION: A graded Lie algebra (Lie superalgebra) is a graded vector space $\mathfrak{g}^*$ equipped with a bilinear graded map $\{\cdot,\cdot\}: \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ which is graded anticommutative: $\{a,b\} = -(-1)^{\tilde{a}\tilde{b}}\{b,a\}$ and satisfies **the super Jacobi identity**

$$\{c,\{a,b\}\} = \{\{c,a\},b\} + (-1)^{\tilde{a}\tilde{c}}\{a,\{c,b\}\}$$

EXAMPLE: Consider the algebra $\operatorname{End}(A^*)$ of operators on a graded vector space, with supercommutator as above. **Then** $\operatorname{End}(A^*), \{\cdot,\cdot\}$ **is a graded Lie algebra.**

Lemma 1: Let d be an odd element of a Lie superalgebra, satisfying $\{d,d\} = 0$, and L an even element. **Then** $\{\{L,d\},d\} = 0$**.**

Proof: $0 = \{L,\{d,d\}\} = \{\{L,d\},d\} + \{d,\{L,d\}\} = 2\{\{L,d\},d\}$. ■

Laplacians and supercommutators

THEOREM: Let

$$\Delta_d := \{d, d^*\}, \quad \Delta_{d^c} := \{d^c, d^{c*}\}, \quad \Delta_\partial := \{\partial, \partial^*\}, \Delta_{\overline{\partial}} := \{\overline{\partial}, \overline{\partial}^*\}.$$

Then $\Delta_d = \Delta_{d^c} = 2\Delta_\partial = 2\Delta_{\overline{\partial}}$. In particular, **$\Delta_d$ preserves the Hodge decomposition.**

Proof: By Kodaira relations, $\{d, d^c\} = 0$, and graded Jacobi identity,

$$\{d, d^*\} = -\{d, \{\Lambda, d^c\}\} = \{\{\Lambda, d\}, d^c\} = \{d^c, d^{c*}\}.$$

Same calculation with $\partial, \overline{\partial}$ gives $\Delta_\partial = \Delta_{\overline{\partial}}$.. Also, $\{\partial, \overline{\partial}^*\} = \sqrt{-1}\,\{\partial, \{\Lambda, \partial\}\} = 0$, (Lemma 1), and the same argument implies that **all anticommutators $\partial, \overline{\partial}^*$, etc. vanish except $\{\partial, \partial^*\}$ and $\{\overline{\partial}, \overline{\partial}^*\}$.** This gives $\Delta_d = \Delta_\partial + \Delta_{\overline{\partial}}$. ■

DEFINITION: The operator $\Delta := \Delta_d$ is called **the Laplacian**.

DEFINITION: Define the Weil operator W_I by $W_I|_{\Lambda^{p,q}} = \sqrt{-1}\,(p-q)\,\mathrm{Id}$. Clearly, $d^c = [W_I, d]$. **The supercommutators of L, Λ, d, W_I generate all differentials and Hodge operators.**

REMARK: We actually proved that **operators L, Λ, d, W_I generate a Lie superalgebra of dimension $(5|4)$ (5 even, 4 odd), with a 1-dimensional center $\mathbb{R}\Delta$.**

Hodge theory

DEFINITION: A differential form η is called **harmonic** if $\Delta\eta = 0$.

REMARK: Let M be a compact Riemannian manifold, and $\mathcal{H}^*(M)$ the space of harmonic forms. For any $\eta \in \mathcal{H}^*(M)$, one has

$$(\eta, \Delta\eta) = (\eta, dd^*\eta + d^*d\eta) = (d\eta, d\eta) + (d^*\eta, d^*\eta) = 0,$$

hence $d\eta = d^*\eta = 0$. Moreover, $(\eta, d\nu) = (d^*\eta, \nu) = 0$, for all $(i-1)$-forms ν. Therefore, **the natural projection from $\mathcal{H}^i(M)$ to $H^i(M)$ is injective.**

DEFINITION: Denote by P the orthogonal projection from $\Lambda^i(M)$ to a finite-dimensional space $\mathcal{H}^*(M)$. This operator is called **the harmonic projection**. Since $(P\eta, d\nu) = 0$, one has $Pd = dP = 0$. Similarly, $Pd^* = d^*P = 0$.

Hodge theory

THEOREM: (Main theorem of Hodge theory)
There is an operator $G_\Delta : \Lambda^i(M) \longrightarrow \Lambda^i(M)$ called **Green operator** which satisfies

$$\Delta \circ G_\Delta = G_\Delta \circ \Delta = \operatorname{Id} - P.$$

Moreover, **G is a compact operator in any of Sobolev's L^2-norms, and maps continuous forms to smooth.**

REMARK: Since $\Delta G_\Delta dx = dx$, and $d\Delta G_\Delta x = dx$, **the Green operator commutes with d and d^*.**

REMARK: One obviously has $x = dd^*x + d^*dx + P(x)$. This gives **a harmonic decomposition** on the space of all smooth forms:

$$\Lambda^i(M) = \operatorname{im} d \perp \operatorname{im} d^* \perp \ker \Delta,$$

Since $\operatorname{im} d^* \perp \ker d$, the space $\ker \Delta$ is therefore identified with cohomology.

dd^c-lemma

THEOREM: (“dd^c-lemma”)
Let M be a compact Kaehler manifold, and $\eta\Lambda^{p,q}(M)$ an exact form. Then $\eta = dd^c\alpha$, for some $\alpha \in \Lambda^{p-1,q-1}(M)$.

Proof. Step 1:
Hodge theory gives $\eta = G_\Delta \Delta\eta$. Since $d\eta = 0$, we obtain

$$\eta = G_\Delta dd^*\eta = dG_\Delta d^*\eta.$$

Step 2:
The same argument also gives $\eta = G_\Delta d^c(d^c)^*\eta = d^cG_\Delta(d^c)^*\eta$.
Comparing these equations, we obtain $\eta = d^cG_\Delta(d^c)^*dG_\Delta d^*\eta$.

Step 3:
Commutator relations (Kodaira) give $\eta = dd^cG_\Delta G_\Delta d^*(d^c)^*\eta$. ■

DEFINITION: A Kaehler class of a Kaehler manifold (M, I, ω) is the cohomology class of ω in $H^{1,1}(M)$. Given two Kaehler forms ω, ω' in the same Kaehler class, one has $\omega - \omega' = dd^c f$, for some function f on M.

Chern connection

Let M be a complex manifold.

DEFINITION: An $(0,1)$-connection on a vector bundle B is an operator $\overline{\partial}: B \longrightarrow B \otimes \Lambda^{0,1}(M)$ which satisfies $\overline{\partial}(fb) = f\overline{\partial}(b) + b \otimes \overline{\partial} f$, for any $f \in C^\infty M$.

REMARK: Using the Leibniz identity $\overline{\partial}(b \otimes \eta) = \overline{\partial}(b) \otimes \eta + b \otimes \overline{\partial}\eta$, we extend any $(0,1)$-connection to the following sequence of maps, also denoted by $\overline{\partial}$

$$B \xrightarrow{\overline{\partial}} B \otimes \Lambda^{0,1}(M) \xrightarrow{\overline{\partial}} B \otimes \Lambda^{0,2}(M) \xrightarrow{\overline{\partial}} B \otimes \Lambda^{0,3}(M) \xrightarrow{\overline{\partial}} ...$$

DEFINITION: A **holomorphic structure** on a vector bundle is a $(0,1)$-connection which satisfies $\overline{\partial}^2 = 0$.

DEFINITION: A **Chern connection** on a holomorphic Hermitian bundle is a Hermitian connection $B \xrightarrow{\nabla} B \otimes \Lambda^1(M)$ which satisfies $\nabla^{0,1} = \overline{\partial}$.

REMARK: The Chern connection exists and is unique. Choose a holomorphic frame $b_1, ..., b_l$ in B (locally), and let h be its Hermitian form. Then $\nabla h = 0$ implies that $\partial h(b_i, \overline{b}_j) = h(\nabla^{1,0} b_i, b_j)$

In particular, when $\dim B = 1$, b a holomorphic section, one has $\nabla^{1,0} b = \frac{\partial b}{|b|} = \partial \log |b|$.

COROLLARY: The curvature of the Chern connection is equal to $dd^c \log |b|$.

Calabi-Yau manifolds

DEFINITION: Let (M, I, ω) be a Kaehler n-manifold, and $K(M) := \Lambda^{n,0}(M)$ its canonical bundle. The natural Hermitian metric on $K(M)$ is written as $(\alpha, \alpha') \longrightarrow \frac{\alpha\wedge\overline{\alpha}'}{\omega^n}$. **Ricci curvature** Ric of M is a curvature of the Chern connection on $K(M)$.

DEFINITION:
A Calabi-Yau manifold is a compact Kaehler manifold with $c_1(M) = 0$.

THEOREM: (Calabi-Yau)
Let (M, I, g) be Calabi-Yau manifold. **Then there exists a unique Ricci-flat Kaehler metric in any given Kaehler class.**

Ricci curvature of Calabi-Yau manifold

REMARK: The canonical bundle of a Calabi-Yau manifold is topologically trivial. Let Φ be a non-degenerate section of $K(M)$, $|\Phi| = 1$. For any metric ν on $K(M)$, the curvature of its Chern connection Θ_ν is equal to $\operatorname{Ric} + dd^c \log|\Phi|_\nu$.

CLAIM: A Kaehler metric ω_1 on M is Ricci-flat iff $dd^c \log \frac{\omega_1^n}{\omega^n} = -\operatorname{Ric}$.

Proof: Let ν be the new metric on $K(M)$ induced by ω_1. Then

$$\Theta_\nu = \operatorname{Ric} + dd^c \log|\Phi|_\nu = \operatorname{Ric} + dd^c \log \frac{\omega_1^n}{\omega^n}.$$

■

REMARK: From dd^c-lemma, we obtain that $\operatorname{Ric} = dd^c f$, for some f.

COROLLARY:
A Kaehler metric ω_1 on M is Ricci-flat iff $\omega_1^n = e^f \omega^n$, where $\operatorname{Ric} = dd^c f$. ■

The Monge-Ampere equation

THEOREM: (Calabi-Yau)
Let (M, ω) be a compact Kaehler n-manifold, and f any smooth function. **Then there exists a unique up to a constant function** φ such that

$$(\omega + dd^c\varphi)^n = Ae^f\omega^n,$$

where A is a positive constant obtained from the formula $\int_M Ae^f\omega^n = \int_M \omega^n$.

REMARK:

$$(\omega + dd^c\varphi)^n = Ae^f\omega^n,$$

is called **the Monge-Ampere equation.**

Uniqueness of solutions of complex Monge-Ampere equation

PROPOSITION: (Calabi) **A complex Monge-Ampere equation has at most one solution,** up to a constant.

Proof. Step 1: Let ω_1, ω_2 be solutions of Monge-Ampere equation. Then $\omega_1^n = \omega_2^n$. By dd^c-lemma, one has $\omega_2 = \omega_1 + dd^c\psi$. **We need to show $\psi = const$.**

Step 2: This gives

$$0 = (\omega_1 + dd^c\psi)^n - \omega_1^n = dd^c\psi \wedge \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}.$$

Step 3: Let $P := \sum_{i=0}^{n-1} \omega_1^i \wedge \omega_2^{n-1-i}$. This is a positive $(n-1, n-1)$-form. **There exists a Hermitian form ω_3 on M such that $\omega_3^{n-1} = P$.**

Step 4: Since $dd^c\psi \wedge P = 0$, this gives $\psi dd^c\psi \wedge P = 0$. Stokes' formula implies

$$0 = \int_M \psi \wedge \partial\overline{\partial}\psi \wedge P = -\int_M \partial\psi \wedge \overline{\partial}\psi \wedge P = -\int_M |\partial\psi|_3^2 \omega_3^n.$$

where $|\cdot|_3$ is the metric associated to ω_3. **Therefore $\overline{\partial}\psi = 0$.** ■