Kähler manifolds and special holonomy,

lecture 4

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Holonomy group

DEFINITION: (Cartan, 1923) Let (B, ∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla}: B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B, ∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: A bundle is **flat** (has vanishing curvature) **if and only if its restricted holonomy vanishes.**

REMARK: If $\nabla(\varphi) = 0$ for some tensor $\varphi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$, **the holonomy group preserves φ.**

EXAMPLE: Holonomy of a Riemannian manifold lies in $O(T_xM, g|_x) = O(n)$.

EXAMPLE: Holonomy of a Kähler manifold lies in $U(T_xM, g|_x, I|_x) = U(n)$.

REMARK: The holonomy group **does not depend on the choice of a point $x \in M$.**

Holonomy representation

DEFINITION: Let (M, g) be a Riemannian manifold, G its holonomy group. A **holonomy representation** is the natural action of G on TM.

THEOREM: Suppose that the holonomy representation is not irreducible: $T_xM = V_1 \oplus V_2$. Then M locally splits as $M = M_1 \times M_2$, with $V_1 = TM_1$, $V_2 = TM_2$.

Proof. Step 1: Using the parallel transform, we extend $V_1 \oplus V_2$ to a **splitting of vector bundles $TM = B_1 \oplus B_2$, preserved by holonomy.**

Step 2: The sub-bundles B_1, $B_2 \subset TM$ **are integrable:** $[B_1, B_1] \subset B_i$ (the Levi-Civita connection is torsion-free)

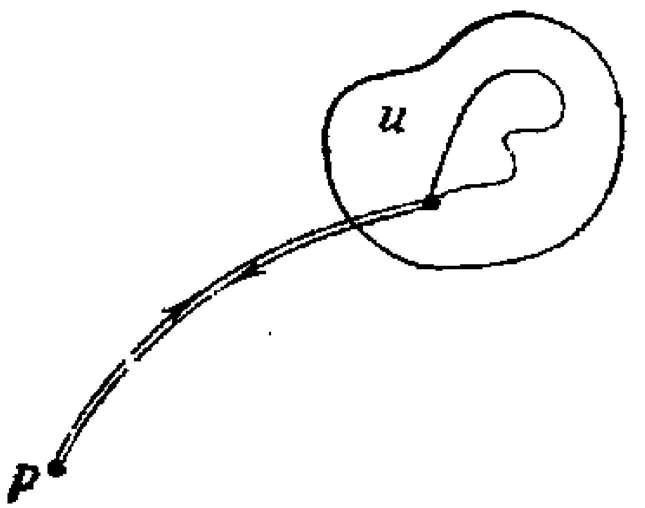
Step 3: Taking integral leaves of these integrable distributions, **we obtain a local decomposition $M = M_1 \times M_2$, with $V_1 = TM_1$, $V_2 = TM_2$.**

Step 4: Since the splitting $TM = B_1 \oplus B_2$ is preserved by the connection, **the leaves M_1, M_2 are totally geodesic.**

Step 5: Therefore, **locally M splits (as a Riemannian manifold)**: $M = M_1 \times M_2$, where M_1, M_2 are any leaves of these foliations. ∎

The Lasso lemma

DEFINITION: A **lasso** is a loop of the following form:



The round part is called **a working part** of a loop.

REMARK: ("The Lasso Lemma") Let $\{U_i\}$ be a covering of a manifold, and γ a loop. Then γ **is a product of several lasso, with working part of each inside some U_i.**

The de Rham splitting theorem

COROLLARY: Let M be a Riemannian manifold, and $\mathcal{H}\mathrm{ol}_0(M) \xrightarrow{\rho} \mathrm{End}(T_xM)$ a reduced holonomy representation. Suppose that ρ is reducible: $T_xM = V_1 \oplus V_2 \oplus ... \oplus V_k$. **Then $G = \mathcal{H}\mathrm{ol}_0(M)$ also splits: $G = G_1 \times G_2 \times ... \times G_k$,** with each G_i acting trivially on all V_j with $j \neq i$.

Proof: Locally, this statement follows from the local splitting of M proven above. To obtain it globally in M, use the Lasso Lemma. ■

THEOREM: (de Rham) A complete, simply connected Riemannian manifold with non-irreducible holonomy **splits as a Riemannian product.**

REMARK: Easy to find non-complete or non-simply connected counterexamples to de Rham theorem.

The Ambrose-Singer theorem

DEFINITION: Let (B, ∇) be a bundle with connection, $\Theta \in \Lambda^2(M) \otimes \mathrm{End}(B)$ its curvature, and $a, b \in T_x M$ tangent vectors. An endomorphism $\Theta(a,b) \in \mathrm{End}(B)|_x$ is called **a curvature element**.

THEOREM: (Ambrose-Singer) The restricted holonomy group of B, ∇ at $z \in M$ is a Lie group, **with its Lie algebra generated by all curvature elements $\Theta(a,b) \in \mathrm{End}(B)|_x$ transported to z along all loops.**

REMARK: Its proof follows from Lasso lemma.

An algebraic holonomy

DEFINITION: Let V be a Euclidean vector space, and $R(V) \subset Sym^2(\Lambda^2 V)$ be the kernel of a multiplication $Sym^2(\Lambda^2 V) \longrightarrow \Lambda^4 V$. Then $R(V)$ is called **the space of algebraic curvature tensors**. We consider it as a subspace in $\Lambda^2(V) \otimes \mathfrak{so}(V)$.

COROLLARY: Let M be a Riemannian manifold, and $\mathfrak{g}$ its holonomy Lie algebra acting on $V = T_x M$. **Then there exists a subspace $S \subset R(V)$ such that $\mathfrak{g} \subset \mathfrak{so}(V)$ is generated (as a Lie algebra) by a subspace $\mu(S \otimes V \otimes V) \subset \mathfrak{so}(V)$,** where

$$\mu : S \otimes V \otimes V \longrightarrow \mathfrak{so}(V)$$

maps Θ, a, b to $\Theta(a, b)$.

DEFINITION: Let $\mathfrak{g} \subset \mathfrak{so}(V)$ be a subalgebra generated by $S \subset R(V)$ as above. Then $\mathfrak{g}$ is called **an algebraic holonomy Lie algebra**.

Berger's theorem and its proof

THEOREM: (Simons, 1962) Let $\mathfrak{g} \subset \mathfrak{so}(V)$ be an algebraic holonomy Lie algebra generated by $S \subset R(V)$. Assume that $\mathfrak{g}$ is irreducible, and $\mathfrak{g}(S) \neq 0$. **Then $G = \operatorname{Lie}(\mathfrak{g})$ acts transitively on the unit sphere in V.**

DEFINITION: A Riemannian manifold M is called **symmetric** if its isometry group $\operatorname{Iso}(M)$ transitively acts on the set of pairs $\{x, y \in M \mid d(x,y) = \varepsilon\}$, for any $\varepsilon \in \mathbb{R}$.

REMARK: This is equivalent to existence of an involution fixing a point $x \in M$ and acting as -1 on T_xM, for each $x \in M$.

DEFINITION: A Riemannian manifold M is called **locally symmetric** if it has a covering which is isometric to an open subset in a symmetric space.

PROPOSITION: **M is locally symmetric if and only if $\nabla(R) = 0$, where R is its Riemannian curvature.**

COROLLARY: (Berger's theorem) Let M be a manifold with irreducible holonomy. **Then either M is locally symmetric, or $\mathcal{H}ol(M)$ acts transitively on the unit sphere in T_xM.**

Berger's list

THEOREM: Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

REMARK: There is one more group acting transitively on a sphere: $Spin(9)$ acting on $\mathbb{R}^{16}$. In 1968, D. Alekseevsky has shown that **a manifold with holonomy $Spin(9)$ is automatically locally symmetric.**

REMARK: A similar list exists for non-orthogonal irreducible holonomy without torsion (Merkulov, Schwachhöfer, 1999).

Spinors and Clifford algebras

DEFINITION: A Clifford algebra of a vector space V with a scalar product q is an algebra generated by V with a relation $xy + yx = q(x, y)1$.

REMARK: A Clifford algebra of a complex vector space with $V = \mathbb{C}^n$ with q non-degenerate **is isomorphic to** Mat$(\mathbb{C}^{n/2})$ **(n even) and** Mat$(\mathbb{C}^{\frac{n-1}{2}}) \oplus$ Mat$(\mathbb{C}^{\frac{n-1}{2}})$ **(n odd).**

DEFINITION: The space of spinors of a complex vector space V, q is a fundamental representation of $Cl(V)$ (n even) and one of two fundamental representations of the components of Mat$(\mathbb{C}^{\frac{n-1}{2}}) \oplus$ Mat$(\mathbb{C}^{\frac{n-1}{2}})$ (n odd).

REMARK: A 2-sheeted covering Spin$(V) \longrightarrow SO(V)$ naturally acts on the spinor space, which is called **the spin representation of Spin(V).**

DEFINITION: Let Γ be a principal $SO(n)$-bundle of a Riemannian oriented manifold M. We say that M is a spin-manifold, **if Γ can be reduced to a $Spin(n)$-bundle.**

REMARK: This happens precisely when the second Stiefel-Whitney class $w_2(M)$ vanishes.

Spinor bundles and Dirac operator

DEFINITION: A bundle of spinors on a spin-manifold M is a vector bundle associated to the principal $Spin(n)$-bundle and a spin representation.

DEFINITION: Consider the map $TM \otimes \mathrm{Spin} \longrightarrow \mathrm{Spin}$ induced by the Clifford multiplication. One defines **the Dirac operator** $D: \mathrm{Spin} \longrightarrow \mathrm{Spin}$ as a composition of $\nabla: \mathrm{Spin} \longrightarrow \Lambda^1 M \otimes \mathrm{Spin} = TM \otimes \mathrm{Spin}$ and the multiplication.

DEFINITION: A **harmonic spinor** is a spinor ψ such that $D(\psi) = 0$.

THEOREM: (Bochner) A harmonic spinor ψ on a compact Ricci-flat manifold satisfies $\nabla\psi = 0$.

Bochner's vanishing on Kaehler manifolds

REMARK: A Kaehler manifold is spin if and only if $c_1(M)$ is even, or, equivalently, if there exists a square root of a canonical bundle $K^{1/2}$.

REMARK: On a Kaehler manifold of complex dimension n, **one has a natural isomorphism between the spinor bundle and $\Lambda^{*,0}(M) \otimes K^{1/2}$** (for n even) and $\Lambda^{2*,0}(M) \otimes K^{1/2}$ (for n odd).

REMARK: On a Kähler manifold, the Dirac operator corresponds to $\partial + \partial^*$.

COROLLARY: On a Ricci-flat Kähler manifold, all $\alpha \in \ker(\partial+\partial^*)\Big|_{\Lambda^{*,0}(M)}$ ara parallel.

THEOREM: (Bochner's vanishing) Let M be a Ricci-flat Kaehler manifold, and $\Omega \in \Lambda^{p,0}(M)$ a holomorphic differential form. **Then $\nabla\Omega = 0$.**

COROLLARY: Let M be a Ricci-flat Kaehler manifold with local holonomy $= SU(n)$. **Then $H^{i,0}(M) = 0$ if $i \neq 0, n$ and $H^{i,0}(M) = \mathbb{C}$ for $i = 0, n$.**

COROLLARY: Let M be a Ricci-flat Kaehler manifold with local holonomy $= Sp(n)$. **Then $H^{i,0}(M) = \mathbb{C}$ if $i = 0, 2, 4, 6, 8, ..., 2n$ and $H^{i,0}(M) = 0$ otherwise.**

Bogomolov's decomposition theorem

THEOREM: (Cheeger-Gromoll) Let M be a compact Riemannian manifold with $\pi_1(M)$ infinite and non-negative Ricci curvature. **Then a universal covering of M is a product of $\mathbb{R}$ and a Ricci-flat manifold.**

THEOREM: (Bogomolov's splitting theorem) Let M be a compact, Ricci-flat Kaehler manifold. **Then there exists a finite covering $\tilde{M}$ of M which is is a product of Kaehler manifolds of the following form:**

$$\tilde{M} = T \times M_1 \times ... \times M_i \times K_1 \times ... \times K_j,$$

with all M_i, K_i simply connected, T a torus, and $\mathcal{H}\text{ol}(M_l) = Sp(n_l)$, $\mathcal{H}\text{ol}(K_l) = SU(m_l)$

REMARK: $\pi_1(M) = 0$ if $\mathcal{H}\text{ol}(M) = Sp(n)$, or $\mathcal{H}\text{ol}(M) = SU(2n)$. If $\mathcal{H}\text{ol}(M) = SU(2n+1)$, $\pi_1(M)$ is finite.