Equivariant projective embeddings

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IMPA,
Estruturas geométricas em variedades,
March 21, 2024

joint with Liviu Ornea

LCK manifolds

DEFINITION: A complex Hermitian manifold (M, I, g, ω) is called **locally conformally Kähler** (LCK) if there exists a closed 1-form θ such that $d\omega = \theta \wedge \omega$. The 1-form θ is called the **Lee form** and the g-dual vector field $\theta^\sharp$ is called the **Lee field**.

REMARK: This definition **is equivalent with the existence of a Kähler cover $(\tilde M, \tilde\omega) \to M$ such that the deck group Γ acts on $(M, \tilde\omega)$ by holomorphic homotheties.** Indeed, suppose that θ is exact, $df = \theta$. **Then $e^{-f}\omega$ is a Kähler form.**

THEOREM: (Vaisman)
A compact LCK manifold with non-exact Lee form **does not admit a Kähler structure.**

REMARK: Such manifold are called **strict LCK**. Further on, **we shall consider only strict LCK manifolds.**

Vaisman manifolds

DEFINITION: The LCK manifold (M, I, g, ω) is a **Vaisman manifold** if the Lee form is parallel with respect to the Levi-Civita connection.

THEOREM: A compact (strictly) LCK manifold M is Vaisman **if and only if it admits a non-trivial action of a complex Lie group of positive dimension,** acting by holomorphic isometries.

DEFINITION: A linear Hopf manifold is a quotient $M := \frac{\mathbb{C}^n \backslash 0}{\langle A \rangle}$ where A is a linear contraction. When A is diagonalizable, M is called **diagonal Hopf.**

CLAIM: All diagonal Hopf manifolds are Vaisman, and **all non-diagonal Hopf manifolds are LCK and not Vaisman**.

THEOREM: A compact complex manifold admits a Vaisman structure **if and only if it admits a holomorphic embedding to a diagonal Hopf manifold.**

THE MAIN QUESTION TODAY: Fix a Vaisman manifold M. **What is the smallest dimension d of a Hopf manifold H such that M can be holomorphically embedded to H? We want to bound d by some polynomial of $\dim M$.**

Algebraic cones

DEFINITION: Let P be a projective orbifold, and L an ample line bundle on P. Assume that the total space $\operatorname{Tot}^\circ(L)$ of all non-zero vectors in L is smooth. **An open algebraic cone** is $\operatorname{Tot}^\circ(L)$.

EXAMPLE: Let $P \subset \mathbb{C}P^n$, and $L = \mathcal{O}(1)|_P$. Then **the open algebraic cone** $\operatorname{Tot}^\circ(L)$ **can be identified with the set** $\pi^{-1}(P)$ of all $v \in \mathbb{C}^{n+1} \backslash 0$ projected to P under the standard map $\pi: \mathbb{C}^{n+1} \backslash 0 \to \mathbb{C}P^n$. **The closed algebraic cone** is its closure in $\mathbb{C}^{n+1}$.

REMARK: The closed algebraic cone **is obtained by adding one point, called "the apex", or "the origin", to** $\operatorname{Tot}^\circ(L)$**.**

THEOREM: Let M be a Vaisman manifold. **Then M admits a $\mathbb{Z}$-cover $\tilde{M}$ which is isomorphic to an open algebraic cone** $\operatorname{Tot}^\circ(L)$**,** associated with an ample line bundle L over a projective orbifold X. Moreover, **the $\mathbb{Z}$-action on $\tilde{M}$ is obtained from an automorphism** $f \in \operatorname{Aut}(X)$ **and an equivariant action of f on L.**

Embedding of Vaisman manifolds to Hopf

THEOREM: Let M be a Vaisman manifold, and $\tilde M = \operatorname{Tot}^\circ(L)$ the corresponding algebraic cone, associated with an ample line bundle L over a projective orbifold X. Assume that the the $\mathbb{Z}$-action on $\tilde M$ is obtained from the linearized automorphism $f \in \operatorname{Aut}(X)$. **Then any embedding of M to a Hopf manifold is obtained from the following data:**

(a) a projective embedding of $X \stackrel{j}{\to} \mathbb{C}P^m$ such that $j^*\mathcal{O}(1) \cong L$

(b) An automorphism $\tilde f$ of $\mathbb{C}P^m$ preserving X and acting on X as f.

Whitney theorem

THEOREM: (Whitney) Let M be a compact smooth manifold, $\dim M = n$. **Then M admits an immersion to $\mathbb{R}^{2n}$ and a smooth embedding to $\mathbb{R}^{2n+1}$.**

Let us prove a projective version of this theorem.
THEOREM: (Whitney) Let M be a projective manifold over $\mathbb{C}$, $\dim_{\mathbb{C}} M = n$. **Then M admits an immersion to $\mathbb{C}P^{2n}$ and a projective embedding to $\mathbb{C}P^{2n+1}$.**

Proof. Step 1: Since M is projective, it can be realized as a subspace in $\mathbb{C}P^m$. It would suffice to show that there is a projection $\mathbb{C}P^m \to \mathbb{C}P^{m-1}$ which induces an injective map on M when $m > 2n+1$ and an immersion when $m > 2n$.

Step 2: Consider a point $x \in \mathbb{C}P^m \backslash M$, and let $\pi_x : \mathbb{C}P^m \backslash \{x\} \to \mathbb{C}P^{m-1}$ be the projection. It is constructed as follows: we identify $\mathbb{C}P^{m-1}$ with the space of lines passing through x, and take $z \in \mathbb{C}P^m \backslash \{x\}$ to the unique line $l_{x,z}$ connecting x and x. **Clearly, $\pi_x : M \to \mathbb{C}P^{m-1}$ is an immersion if and only if $l_{x,z}$ is not tangent to M for any $z \in M$**, and injective if and only if $l_{x,z} \cap M = \{z\}$ (that is, **$\pi_x : M \to \mathbb{C}P^{m-1}$ is injective if any line passing through x intersects M in at most one point).**

Whitney theorem (2)

Step 3: Consider the set $\mathcal{T}$ of all lines tangent to some $z \in M$, and the set $\mathcal{S}$ of all lines connecting two distinct points $z_1 \neq z_2 \in M$. Denote by $L(\mathcal{T}), L(\mathfrak{S}) \subset \mathbb{C}P^m$ the union of all lines in $\mathcal{T}$, $\mathfrak{S}$. From Step 2 it is clear that $\pi_x: M \to \mathbb{C}P^{m-1}$ **is injective if $x \notin L(\mathfrak{S})$, and is immersion if $x \notin L(\mathfrak{T})$.**

Step 4: The dimension of $\mathcal{T}$ is $\leqslant 2n-1$, because it is an image of $\mathbb{P}TM$, and $\dim \mathcal{S} \leqslant 2n$, because it is an image of $M \times M \backslash \Delta$, where Δ is diagonal. On the other hand, $L(Z) \leqslant \dim Z + 1$, because it is an image of a family of lines parametrized by Z. This implies that $\dim L(\mathcal{S}) \leqslant 2n+1$ and $\dim L(\mathcal{T}) \leqslant 2n$, hence $\mathbb{C}P^m \backslash L(\mathcal{T})$ is non-empty when $m > 2n$ and $\mathbb{C}P^m \backslash L(\mathcal{S})$ is non-empty when $m > 2n+1$. **Choosing $x \in \mathbb{C}P^m \backslash L(\mathcal{T})$ or $x \in \mathbb{C}P^m \backslash L(\mathcal{S})$ and using Step 3, we obtain that $\pi_x: M \to \mathbb{C}P^{m-1}$ is an immersion (embedding).** ■

Equivariant projective embeddings

DEFINITION: Let f be an automorphism of a projective manifold X. The automorphism is called **linearizable** if it acts equivariantly on an ample bundle.
EXERCISE: Prove that f **is linearizable if and only if there exists a projective embedding** $X \hookrightarrow \mathbb{C}P^m$ **and** $\tilde{f} \in PGL(m+1,\mathbb{C}) = \operatorname{Aut}(\mathbb{C}P^m)$ **such that** $\tilde{f}$ **preserves** X **and** $\tilde{f}|_X = f$**.**

DEFINITION: Let $f \in \operatorname{Aut}(X)$, where X is a projective variety A projective embedding $X \hookrightarrow \mathbb{C}P^m$ is called **f-equivariant** if f can be extended to $\tilde{f} \in PGL(m+1,\mathbb{C}) = \operatorname{Aut}(\mathbb{C}P^m)$.

We are interested in the following problem.
QUESTION: Consider an n-dimensional projective manifold X. **What is the minimal m such that f-equivariant embedding exists?** We want m to be a function of n, like in Whitney's theorem.

REMARK: Define **a projection** as a composition of a sequence of projections with a center in a point. Consider all projections $\pi: \mathbb{C}P^m \to \mathbb{C}P^{m_1}$ such that $\pi: X \to \pi(X)$ is an isomorphism. Following the same logic as in the proof of Whitney's theorem, **we want to find the smallest m_1 such that there exists $f_1 \in PGL(m_1+1,\mathbb{C})$ preserving the image $\pi(X)$ of X and acting on $\pi(X)$ as f.** This problem can be understood as **"the equivariant Whitney embedding problem"**.

Projections with centers of any dimension

REMARK 1: Let $U, V \subset W$ be non-intersecting subspaces of a vector space, $\dim U + \dim V = \dim W$. Clearly, for any point $x \in W$, there exists unique $u \in U, v \in V$ such that $x = u + v$. Therefore, **for any $x \in \mathbb{P}W \backslash (\mathbb{P}U \cap \mathbb{P}V)$ there exists a unique line l_x which intersects $\mathbb{P}U$ and $\mathbb{P}V$.**

DEFINITION: Let $X, Y \subset \mathbb{P}^m$ be projective subspaces, $\dim X + \dim Y = m-1$. For any $z \in \mathbb{P}^m \backslash (Y \cup X)$, consider the line l_z connecting z, X and Y; it is unique, as follows from Remark 1. **Projection π_Y with center in Y** takes z to the point $l_z \cap X$. When $z \in X$, we take $\pi_Y(z) = z$;

REMARK: It is not hard to see that the map $\pi_Y : \mathbb{P}^m \backslash Y \to X$, defined this way, is holomorphic. Also, **π_Y can be obtained by taking a composition of projections from points $y_1, ..., y_k$, such that $Y = \mathbb{P}\langle y_1, ..., y_k \rangle$, and $\dim Y = k-1$.**

Linear systems and projective embeddings

DEFINITION: Let L be a holomorphic line bundle on a projective variety X. A **linear system of divisors** is a subspace $W \subset H^0(X,L)$. For any $w \in \mathbb{P}(H^0(X,L))$, denote by D_v the zero divisor of v. The line system is the family D_v of divisors, where v runs through $H^0(X,L)$.

PROPOSITION: Projective embeddings $X \subset \mathbb{C}P^m$ **are in bijective correspondence with line systems $W \subset H^0(X,L)$, where L is a very ample line bundle on X, and the following two conditions are satisfied.**

(a) **"Sections separate points":** for every two points $x \neq y \in X$, there exists $a, b \in W$ such that $a|_x = b|_y = 0$ and $a|_y \neq 0, b|_x \neq 0$.

(b) For every $x \in X$, there exists a collaction $a_i \in W$ such that each a_i has a simple zero in x (that is, $a|_x = 0$ and $da|_x \neq 0$), and the differentials da_i generate $T^*_x X$.

Proof: Next slide.

REMARK: These are **the same conditions which guarantee that a very ample bundle L defines a projective embedding.**

Linear systems and projective embeddings (2)

PROPOSITION: Projective embeddings $X \subset \mathbb{C}P^m$ **are in bijective correspondence with line systems $W \subset H^0(X, L)$, where L is a very ample line bundle on X, and the following two conditions are satisfied.**

(a) **"Sections separate points":** for every two points $x \neq y \in X$, there exists $a, b \in W$ such that $a|_x = b|_y = 0$ and $a|_y \neq 0, b|_x \neq 0$.

(b) For every $x \in X$, there exists a collaction $a_i \in W$ such that each a_i has a simple zero in x (that is, $a|_x = 0$ and $da|_x \neq 0$), and the differentials da_i generate $T_x^* X$.

Proof. Step 1: For any $x \in X$, consider the restriction map $\lambda_x : W \to L|_x$, which is a complex linear map from a vector space to a line. Condition (a) implies that $\lambda_x \neq 0$, hence this map defines a point in $\mathbb{P}W^*$. We obtain a holomorphic map $\varphi : X \to \mathbb{P}W^*$. **Condition (a) implies that φ is injective, condition (b) that its derivative is non-zero, hence φ is also an immersion.**

Step 2: Conversely, consider an embedding $j : X \hookrightarrow \mathbb{P}W^*$. Then $L := j^*(\mathcal{O}(1))$ is very ample, and the map $j^* : H^0(\mathcal{O}(1)) \to H^0(X, L)$ is surjective; **its image is a linear system of divisors, which satisfy (a) and (b) because j is injective and immersive.** ■

Linear systems and projective embeddings (3)

REMARK: The correspondence between the projection $\pi_U : \mathbb{C}P^m \to \mathbb{C}P^r$ with the center in $\mathbb{P}U \subset \mathbb{C}P^m$ and the linear systems is given as follows. Let $W \subset H^0(\mathbb{C}P^m, \mathcal{O}(1))$ be the subspace consisting of all sections vanishing in $\mathbb{P}U$, and $X \subset \mathbb{C}P^m$ a projective subvariety disjoint from $\mathbb{P}U$. **Then $\pi_U : X \to \mathbb{C}P^r = \mathbb{P}W^*$ is the projective embedding associated with the linear system $W \subset H^0(\mathbb{C}P^m, \mathcal{O}(1))$.**

COROLLARY: Let $f \in \operatorname{Aut}(X)$ be an automorphism of a projective manifold, and $j : X \hookrightarrow \mathbb{C}P^m = \mathbb{P}V$ a projective embedding. We assume that j is f-equivariant, that is, there exists $\tilde{f} \in PGL(m+1, \mathbb{C})$ preserving X and acting on X as f. Consider a projection $\pi : \mathbb{C}P^m \backslash \mathbb{P}U \to \mathbb{C}P^r$ with center in $\mathbb{P}U$ inducing an embedding $\pi_U : X \to \mathbb{C}P^r$. Let $W \subset H^0(\mathbb{C}P^m, \mathcal{O}(1))$ be a subspace consisting of all sections vanishing in $\mathbb{P}U \subset \mathbb{P}V$, and $W_f \subset H^0(\mathbb{C}P^m, \mathcal{O}(1))$ any $\tilde{f}$-invariant space containing W. **Then the projection $\pi : X \to \mathbb{C}P^r = \mathbb{P}W_f^*$ associated with this linear system is f-equivariant.** ■

In other words, to construct an optimal f-equivariant projective embedding, **we need to bound the dimension of the smallest $\tilde{f}$-invariant space W_f containing W.** Here, **the dimension of W can be chosen as low as $\dim W = 2\dim X + 2$ by the projective Whitney theorem.**

Equivariant Whitney theorem

CONJECTURE: After passing to a finite quotient of X, and replacing f by its finite power, we would have an estimate $\dim W_f \leqslant d \dim W$, where d is the dimension of the Zariski closure of the group $\langle f \rangle$.

REMARK: It is not hard to prove that $d \leqslant n$.

If this conjecture is true, we can conclude with the following theorem.

Theorem (modulo the conjecture above):
Let X be an n-dimensional projective manifold, and $f \in \operatorname{Aut}(X)$ a linearizable automorphism. Denote by G_f the Zariski closure of $\langle f \rangle$, and let $\dim G_f = d$. Then **there exists an integer $k > 0$ and a finite group Γ acting on X and commuting with f^k, such that X/Γ admits an f^k-equivariant projective embedding to $\mathbb{C}P^r$, where $r \leqslant d(2n+2) - 1$.**