dd^c-lemma for manifolds with parallel differential forms

"Geometric structures on manifolds", HSE

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Massey products

Let $a, b, c \in \Lambda^*(M)$ be closed forms on a manifold M with cohomology classes $[a], [b], [c]$ satisfying $[a][b] = [b][c] = 0$, and $\alpha, \gamma \in \Lambda^*(M)$ forms which satisfy $d(\alpha) = a \wedge b$, $d(\gamma) = b \wedge c$. Denote by $L_{[a]}, L_{[c]} : H^*(M) \longrightarrow H^*(M)$ the operation of multiplication by the cohomology classes $[a], [c]$.

Then $\alpha \wedge c - (-1)^{\tilde{a}} a \wedge \gamma$ is a closed form, and its cohomology class is well-defined modulo $\operatorname{im} L_{[a]} + \operatorname{im} L_{[c]}$.

DEFINITION: Cohomology class $\alpha \wedge c - (-1)^{\tilde{a}} a \wedge \gamma$ is called **Massey product of a, b, c.**

Heisenberg group

DEFINITION: The **Heisenberg group** G group of strictly upper triangular matrices (3x3),

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

The integer Heisenberg group $G_{\mathbb{Z}}$ is the same group with integer entries. The **Heisenberg nilmanifold** is $G/G_{\mathbb{Z}}$. The Heisenberg nilmanifold **is fibered over the torus T^2 with the fiber S^1** (it is a non-trivial principal S^1-bundle). This fibration corresponds to the exact sequence

$$\{e\} \longrightarrow \mathbb{Z} \longrightarrow G_{\mathbb{Z}} \longrightarrow \mathbb{Z}^2 \longrightarrow \{e\}$$

where $\mathbb{Z}$ is the center.

Massey products in Heisenberg nilmanifold

CLAIM: Masey products on $G/G_{\mathbb{Z}}$ are non-zero.

Proof. Step 1: G acts on $\Lambda^*(G)$ from the right. It is not hard to see that the all cohomology classes on $G/G_{\mathbb{Z}}$ can be represented by right G-invariant forms, and, moreover, **the cohomology of $G/G_{\mathbb{Z}}$ is equal to the cohomology of the complex of right-G-invariant forms on G.**

Step 2: This is the same complex as **the Chevalley-Eilenberg complex** for the Lie algebra $\mathfrak{g}$ of G: $0 \longrightarrow \Lambda^1(\mathfrak{g}^*) \xrightarrow{d} \Lambda^2(\mathfrak{g}^*) \xrightarrow{d}$ ... with the differential in the first term $d: \mathfrak{g}^* \longrightarrow \Lambda^2(\mathfrak{g}^*)$ dual to the commutator. We extend this differential to $\Lambda^*(\mathfrak{g}^*)$ by the Leibniz rule. The corresponding cohomology is called **the Lie algebra cohomology** and denoted by $H^*(\mathfrak{g})$.

Step 3: Let a, b, t be the basis in $\mathfrak{g}$, with the only non-trivial commutator $[a, b] = t$, and α, β, τ the dual basis in $\mathfrak{g}^*$, with the only non-trivial differential $d\tau = \alpha \wedge \beta$. This gives a basis $\alpha \wedge \beta$, $\alpha \wedge \tau$, $\beta \wedge \tau$ in $\Lambda^2(\mathfrak{g}^*)$, with $d\big|_{\Lambda^2\mathfrak{g}^*} = 0$, giving $\operatorname{rk} H^1(G/G_{\mathbb{Z}}) = 2$ and $\operatorname{rk} H^2(G/G_{\mathbb{Z}}) = 2$.

Step 4: Let $M(\alpha, \beta, \alpha)$ denote the Massey product of α, β, α. Since $\alpha\wedge\beta = d\tau$, $M(\alpha, \beta, \alpha) = \tau \wedge \alpha - \alpha \wedge \tau = 2\tau \wedge \alpha$. The image of $L_\alpha: H^1(\mathfrak{g}) \longrightarrow H^2(\mathfrak{g})$ is generated by $\alpha \wedge \beta$, **hence $M(\alpha, \beta, \alpha)$ is non-zero modulo $\operatorname{im} L_\alpha$.** ■

Quasi-isomorphism of DG-algebras

DEFINITION: Graded commutative algebra, or **super-commutative algebra** is a graded algebra $A^* = \bigoplus_{i \geqslant 0} A^i$, with the multiplication $A^i \cdot A^j \longrightarrow A^{i+j}$ and graded (super-) commutativity relation $x \cdot y = (-1)^{\tilde{x}\tilde{y}} y \cdot x$, where $x \in A^{\tilde{x}}, y \in A^{\tilde{y}}$. **A DG-algebra (differential graded algebra)** is a graded commutative algebra equipped with a differential $d: A^i \longrightarrow A^{i+1}$ which satisfies $d(x \cdot y) = dx \cdot y + (-1)^{\tilde{x}} x \cdot dy$. A homomorphism of DG-algebras is called **a quasi-isomorphism** if it induces an isomorphism on cohomology. Two DG-algebras are called **quasi-isomorphic** if they can be connected by a chain of quasi-isomorphisms.

DEFINITION: A DG-algebra is called **formal** if it is quasi-isomorphic to its cohomology algebra (with zero differential). A manifold is called **formal** if its de Rham algebra is formal.

REMARK: A formal algebra has vanishing Massey products, hence the Heisenbern nil-manifold is not formal.

REMARK: The converse is not true, however, there exists a way to define **"the generalized Massey products"** in terms of Maurer-Cartan equation, and their vanishing is equivalent to formality.

Hodge theory

DEFINITION: Let (M,I) be a complex manifold, $\{U_i\}$ its covering, and and $z_1, ..., z_n$ holomorphic coordinate system on each covering patch. **The bundle $\Lambda^{p,q}(M,I)$ of (p,q)-forms on (M,I)** is generated locally on each coordinate patch by monomials $dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p} \wedge d\overline{z}_{i_{p+1}} \wedge ... \wedge dz_{i_{p+q}}$. **The Hodge decomposition** is a decomposition of vector bundles:

$$\Lambda^d_{\mathbb{C}}(M) = \bigoplus_{p+q=d} \Lambda^{p,q}(M).$$

DEFINITION: A manifold is called **Kähler** if it equipped with a closed real (1,1)-form ω such that $\omega(Ix,x) > 0$ for any non-zero vector x.

THEOREM: ("Hodge decomposition on cohomology") Let M be a compact Kähler manifold. **Then any cohomology class can be represented as a sum of closed (p,q)-forms.**

dd^c-lemma

DEFINITION: Let M be a complex manifold, and $I: TM \longrightarrow TM$ its complex structure operator. **The twisted differential** of M is $IdI^{-1}: \Lambda^*(M) \longrightarrow \Lambda^{*+1}(M)$, where I acts on 1-forms as an operator dual to $I: TM \longrightarrow TM$, and on the rest of differential forms multiplicatively.

THEOREM: Let η be a form on a compact Kähler manifold, satisfying one of the following conditions.
(1). η is an exact (p,q)-form. (2). η is d-exact, d^c-closed.
Then $\eta \in \operatorname{im} dd^c$.

COROLLARY 1: Let $(\Lambda^*_{d^c-cl}(M), d)$ be the algebra of d^c-closed forms on a compact Kähler manifold M, considered as a DG-algebra. **Then the natural embedding $\Psi: (\Lambda^*_{d^c}(M), d) \longrightarrow (\Lambda^*(M), d)$ is a quasi-isomorphism.**

Proof: Any closed (p,q)-form is d^c-closed. Since any cohomology class can be represented as a sum of closed (p,q)-forms, it can be represented by d^c-closed form. Therefore Ψ is surjective on cohomology.

Suppose that Ψ maps a d-closed form $\alpha \in \Lambda^*_{d^c}(M)$ to an exact form. By dd^c-lemma, $\alpha = dd^c\beta$, hence $\alpha \in d\left(\Lambda^*_{d^c}(M)\right)$. ■

Formality for Kähler manifolds

THEOREM: A compact Kähler manifold is formal.

Proof. Step 1: The natural embedding $(\Lambda^*_{d^c}(M), d) \longrightarrow (\Lambda^*(M), d)$ is a quasi-isomorphism (Corollary 1).

Step 2: Let $\Phi : (\Lambda^*_{d^c}(M), d) \longrightarrow (H^*_{d^c}(M), 0)$ map a d^c-closed form to its cohomology class in $\frac{\ker d^c}{\operatorname{im} d^c} =: H^*_{d^c}(M)$. Representing cohomology classes by closed (p, q)-forms, we find that $d\Big|_{H^*_{d^c}(M)} = 0$. Therefore, Φ is a homomorphism of DG-algebras.

It is surjective on cohomology because each class in $H^*_{d^c}(M)$ can be represented by a d, d^c-closed form, and injective because each d^c-closed, d-exact form is dd^c-exact. ■

Holonomy group

DEFINITION: (Cartan, 1923) Let (B, ∇) be a vector bundle with connection over M. For each loop γ based in $x \in M$, let $V_{\gamma,\nabla}: \ B|_x \longrightarrow B|_x$ be the corresponding parallel transport along the connection. The **holonomy group** of (B, ∇) is a group generated by $V_{\gamma,\nabla}$, for all loops γ. If one takes all contractible loops instead, $V_{\gamma,\nabla}$ generates **the local holonomy**, or **the restricted holonomy** group.

REMARK: A bundle is **flat** (has vanishing curvature) **if and only if its restricted holonomy vanishes.**

REMARK: If $\nabla(\varphi) = 0$ for some tensor $\varphi \in B^{\otimes i} \otimes (B^*)^{\otimes j}$, **the holonomy group preserves φ.**

DEFINITION: **Holonomy of a Riemannian manifold** is holonomy of its Levi-Civita connection.

EXAMPLE: Holonomy of a Riemannian manifold lies in $O(T_xM, g|_x) = O(n)$.

EXAMPLE: Holonomy of a Kähler manifold lies in $U(T_xM, g|_x, I|_x) = U(n)$.

REMARK: The holonomy group **does not depend on the choice of a point $x \in M$.**

Berger's list

THEOREM: (Berger's theorem, 1955)
Let G be an irreducible holonomy group of a Riemannian manifold which is not locally symmetric. **Then G belongs to the Berger's list:**

Berger's list	
Holonomy	*Geometry*
$SO(n)$ acting on $\mathbb{R}^n$	Riemannian manifolds
$U(n)$ acting on $\mathbb{R}^{2n}$	Kähler manifolds
$SU(n)$ acting on $\mathbb{R}^{2n}$, $n > 2$	Calabi-Yau manifolds
$Sp(n)$ acting on $\mathbb{R}^{4n}$	hyperkähler manifolds
$Sp(n) \times Sp(1)/\{\pm 1\}$ acting on $\mathbb{R}^{4n}$, $n > 1$	quaternionic-Kähler manifolds
G_2 acting on $\mathbb{R}^7$	G_2-manifolds
$Spin(7)$ acting on $\mathbb{R}^8$	$Spin(7)$-manifolds

REMARK: If the holonomy group is not irreducible, the manifold M **locally splits onto a product of Riemannian manifolds with irreducible holonomy** (de Rham). **This splitting is global,** if M is complete and simply connected.

The structure operator

A C^∞-linear map $\Lambda^1(M) \xrightarrow{p} \Lambda^i(M)$ can be uniquely extended to a C^∞-linear derivation ρ on $\Lambda^*(M)$, using the rule

$$\rho\Big|_{\Lambda^1(M)} = p, \quad \rho\Big|_{\Lambda^0(M)} = 0, \quad \rho(\alpha\wedge\beta) = \rho(\alpha)\wedge\beta + (-1)^{(i-1)\tilde\alpha}\alpha\wedge\rho(\beta).$$

Then, ρ is an even for odd p and odd for even p differentiation of the graded commutative algebra $\Lambda^*(M)$.

DEFINITION: Let M be a Riemannian manifold, and $\omega\in\Lambda^k(M)$ a differential form. Consider an operator $C:\ \Lambda^1(M)\longrightarrow\Lambda^{k-1}(M)$ mapping $\nu\in\Lambda^1(M)$ to $\omega\lrcorner\nu^\sharp$, where $\nu^\sharp$ is the vector field dual to ν. Alternatively, $C(\nu)$ can be written as $C(\nu) = *(*\omega\wedge\nu)$. The corresponding differentiation

$$C:\ \Lambda^*(M)\longrightarrow\Lambda^{*+k-2}(M)$$

is called **the structure operator of (M,ω)**. Parity of C is equal to that of ω.

REMARK: If ω is the Kähler form on a Kähler manifold, we have $C = I$.

Twisted de Rham operator

DEFINITION: Let M be a Riemannian manifold, and $\omega \in \Lambda^k(M)$ a differential form, which is parallel with respect to the Levi-Civita connection. Denote by d_c the supercommutator

$$\{d, C\} := dC - (-1)^{\tilde{C}} Cd$$

This operator is called **the twisted de Rham operator of (M, ω)**. Being a graded commutator of two graded differentiations, d_c is also a graded differentiation of $\Lambda^*(M)$.

REMARK: On a Kähler manifold (M, ω), d_c **is the usual twisted de Rham operator $d^c = IdI^{-1}$.** However, for a general form ω, d_c^2 **can be non-zero.**

PROPOSITION: Let (M, ω) be a Riemannian manifold equipped with a parallel form ω, and L_ω the operator $\eta \longrightarrow \eta \wedge \omega$. **Then $d_c = \{L_\omega, d^*\}$, where $\{\cdot, \cdot\}$ denotes the supercommutator,**

$$\{L_\omega, d^*\} = L_\omega d^* - (-1)^{\tilde{\omega}} d^* L_\omega,$$

and $d^* = - * d *$ is the adjoint to d.

Proof: Using the standard arguments, we check that both d_c and $\{L_\omega, d^*\}$ are derivations on the de Rham algebra, and then check the equality $d_c = \{L_\omega, d^*\}$ on 1-forms. ∎

Harmonic forms

COROLLARY: The twisted de Rham operator and formality **commutes with d and d^*.**

Proof: Use **"the basic lemma"**: Let δ be an odd element in a graded Lie superalgebra A satisfying $\{\delta, \delta\} = 0$. Then $\{\delta, \{\delta, x\}\} = 0$ for all $x \in A$, assuming that the base field is not of characteristic 2. Indeed, the graded Jacobi identity gives

$$\{\delta, \{\delta, x\}\} = -\{\delta, \{\delta, x\}\} + \{\{\delta, \delta\}, x\},$$

hence $2\{\delta, \{\delta, x\}\} = 0$. ■

Let M be a compact Riemannian manifold. Using the Hodge theory, **we obtain the following direct sum decomposition:** $\ker d = \operatorname{im} d \oplus \ker d \cap \ker d^*$**.** Indeed, $\ker d^* = (\operatorname{im} d)^\perp$ (this is true whenether $\operatorname{im} d$ is closed, for $\Lambda^*(M)$ the closedness of $\operatorname{im} d$ is implied the Hodge theory).

DEFINITION: A form on a compact Riemannian manifold is called **harmonic** if it lies in $\ker d \cap \ker d^*$; the space of harmonic forms is identified with the cohomology.

Twisted de Rham operator and formality

LEMMA: The twisted differential d_c vanishes on harmonic forms.

Proof: L_ω maps harmonic form to harmonic forms **(this is a non-trivial theorem),** hence $\{d^*, L_\omega\}$ acts on harmonic forms trivially. ■

PROPOSITION: Let (M, ω) be a compact Riemannian manifold equipped with a parallel form. **Then the natural embedding $\Psi : (\ker d_c, d) \hookrightarrow (\Lambda^*(M), d)$. is a quasi-isomorphism.**

Proof: Since d_c vanishes on harmonic forms, and they represent the cohomology, Ψ is surjective on cohomology.

To prove injectivity, assume that $\Psi(\alpha)$ is exact for a d_c, d-closed form. This means that α is d_c-closed and d-exact. We need to show that $\alpha = d(d^c$-closed form).

Let G_Δ be the Green operator inverting the Laplacian $\Delta = dd^* + d^*d$ on forms which are orthogonal to its kernel. Then $\alpha = dd^*G_\Delta(\alpha)$. However, $d_c(d^*G_\Delta\alpha) = 0$ because d_c commutes with G_δ and with d^*. ■