Cousin groups and generalized Oeljeklaus-Toma manifolds

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Most Cited Publications

Citations	Publication
62	**MR0514769 (80c:32024)** Bogomolov, F. A. Hamiltonian Kählerian manifolds. (Russian) *Dokl. Akad. Nauk SSSR* **243** (1978), no. 5, 1101–1104. (Reviewer: R. R. Simha) 32G05 (32J99 53C55)
62	**MR0522939 (80j:14014)** Bogomolov, F. A. Holomorphic tensors and vector bundles on projective manifolds. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **42** (1978), no. 6, 1227–1287, 1439. (Reviewer: R. R. Simha) 14F05 (14D25 14L15 32J15 32L05)
50	**MR0574307 (81c:14025)** Bogomolov, Fedor Alekseivich Sur l'algébricité des représentations l-adiques. (French) *C. R. Acad. Sci. Paris Sér. A-B* **290** (1980), no. 15, A701–A703. 14K15 (14G25)
45	**MR0903621 (88m:16006)** Bogomolov, F. A. The Brauer group of quotient spaces of linear representations. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **51** (1987), no. 3, 485--516, 688; *translation in* Math. USSR-Izv. **30** (1988), no. 3, 455–485 (Reviewer: Jan van Geel) 16A16 (12F20 13A20 14F20)
36	**MR1823313 (2002g:57051)** Amorós, J.; Bogomolov, F.; Katzarkov, L.; Pantev, T. Symplectic Lefschetz fibrations with arbitrary fundamental groups. With an appendix by Ivan Smith. *J. Differential Geom.* 54 (2000), no. 3, 489–545. (Reviewer: Margaret F. Symington) 57R17 (53D35)
35	**MR1797587 (2002b:14025)** Bogomolov, F. A.; Tschinkel, Yu. Density of rational points on elliptic $K3$ surfaces. *Asian J. Math.* 4 (2000), no. 2, 351–368. (Reviewer: Matthew H. Baker) 14G05 (11G05 11G35 14C22 14J10 14J28)
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24	**MR1397679 (97e:14024)** Bogomolov, Fedor A.; Pantev, Tony G. Weak Hironaka theorem. *Math. Res. Lett.* 3 (1996), no. 3, 299–307. (Reviewer: Yuri G. Prokhorov) 14E15 (14B05 14E05)
24	**MR1695791 (2000e:14025)** Bogomolov, F. A.; Tschinkel, Yu. On the density of rational points on elliptic fibrations. *J. Reine Angew. Math.* 511 (1999), 87–93. (Reviewer: Constantin D. Manoil) 14G05 (11G35 14J45)
23	**MR0788089 (86k:14033)** Bogomolov, F. A.; Katsylo, P. I. Rationality of some quotient varieties. (Russian) *Mat. Sb. (N.S.)* 126(168) (1985), no. 4, 584–589. (Reviewer: I. Dolgachev) 14L30 (14E05 20G05)
19	**MR0587337 (81m:14031)** Bogomolov, F. A. Points of finite order on abelian varieties. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **44** (1980), no. 4, 782–804, 973. (Reviewer: Reinhard Bölling) 14K15 (14G25)
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17	**MR1462922 (99a:32015)** Bogomolov, Fedor A.; de Oliveira, Bruno Stein small deformations of strictly pseudoconvex surfaces. *Birational algebraic geometry (Baltimore, MD, 1996)*, 25–41, Contemp. Math., 207, *Amer. Math. Soc., Providence, RI,* 1997. (Reviewer: S. Dimiev) 32G05 (14C25 32F30)
16	**MR1406665 (98b:32025)** Bogomolov, F. A. On the cohomology ring of a simple hyper-Kähler manifold (on the results of Verbitsky). *Geom. Funct. Anal.* 6 (1996), no. 4, 612–618. (Reviewer: Andrew Swann) 32J27 (32G05 32G20 53C25)
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Solvmanifolds

DEFINITION: Let M be a smooth manifold equipped with a transitive action of solvable Lie group. Then M is called **a solvmanifold**.

REMARK: All solvmanifolds are obtained as quotient spaces, $M = G/H$.

DEFINITION: An integrable complex structure on a real Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{g}^{1,0} \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ such that $\mathfrak{g}^{1,0} \oplus \overline{\mathfrak{g}^{1,0}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$

REMARK: Right-invariant complex structures on a connected real Lie group **are in 1 to 1 correspondence with integrable complex structures** on its Lie algebra.

DEFINITION: A **complex solvmanifold** is a solvmanifold $M = G/H$ equipped with a complex structure, in such a way that G has a right-invariant complex structure, and the projection $G \longrightarrow M$ is holomorphic.

REMARK: Solvmanifolds are usually non-homogeneous (as complex manifolds).

Inoue surfaces

DEFINITION: ("Bogomolov's theorem") **Inoue surface** is a complex surface without curves and with $b_2 = 0$.

REMARK: Original definiton of Inoue was constructive, in terms of explicit action by matrices, and the above result is a theorem proven by Bogomolov in 1976.

Bogomolov, F. A. *Classification of surfaces of class VII_0 with $b_2 = 0$*. Izv. Akad. Nauk SSSR Ser. Mat. 40 (1976), no. 2, 273-288, 469.

СЕРИЯ
МАТЕМАТИЧЕСКАЯ
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Ф. А. БОГОМОЛОВ

КЛАССИФИКАЦИЯ ПОВЕРХНОСТЕЙ КЛАССА VII_0 с $b_2=0$

Введение

Класс поверхностей VII_0 — поверхностей, которые по своим топологическим свойствам отличаются от кэлеровых, был впервые введен K. Kodaira. Им же была поставлена проблема классификации таких поверхностей.

History of Inoue surfaces

In 1991, a new proof appeared, based on Yang-Mills theory:
Li, J.; Yau, S.-T.; Zheng, F. *A simple proof of Bogomolov's theorem on class* VII_0 *surfaces with* $b_2 = 0$. Illinois J. Math. 34 (1990), no. 2, 217-220.

This proof was incorrect.

Finally, correct proofs were also obtained.

Teleman, Andrei Dumitru *Projectively flat surfaces and Bogomolov's theorem on class* VII_0 *surfaces.* Internat. J. Math. 5 (1994), no. 2, 253-264.

Li, Jun; Yau, Shing-Tung; Zheng, Fangyang *On projectively flat Hermitian manifolds.* Comm. Anal. Geom. 2 (1994), no. 1, 103-109.

Complex solvmanifolds of dimension 2

THEOREM: (Hasegawa) Let M be a complex surface which is diffeomorphic to a solvmanifold. Then M is (up to a finite unramified quotient) **isomorpic to one the following.**

1. Compact complex torus

2. Kodaira surface

3. Inoue surface.

This theorem directly follows from Bogomolov's theorem, Hasegawa's result on Kähler solvmanifolds, and Kodaira's classification.

To define the Inoue surfaces explicitly, we use **the number theory**.

Dirichlet unit theorem

DEFINITION: Let $K:\mathbb{Q}$ be a number field of degree n. **The ring of integers** $\mathcal{O}_K \subset K$ is an integral closure of $\mathbb{Z}$ in K, that is, the set of all roots in K of monic polynomials $P(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + ... + a_0$ with integer coefficients $a_i \in \mathbb{Z}$.

CLAIM: An additive group $\mathcal{O}_K^+$ is a finitely generated abelian group of rank n.

DEFINITION: A unit of a ring $\mathcal{O}_K$ is an element $u \in \mathcal{O}_K$, such that u^{-1} also belongs to $\mathcal{O}_K$.

Dirichlet's unit theorem: Let K be a number field which has s real embeddings and $2t$ complex ones. Then **the group of units $\mathcal{O}_K^*$ is isomorphic to $G \times \mathbb{Z}^{t+s-1}$,** where G is a finite group of roots of unity contained in K. Moreover, if $s > 0$, one has $G = \pm 1$.

REMARK: For an imaginary quadratic field, **the group of units is a group of solutions of Pell's equation.**

Cubic fields and complex surfaces

Let $K:\mathbb{Q}$ be a cubic extension of $\mathbb{Q}$ which has 2 complex embeddings τ, $\overline{\tau}$ and one real one σ (such an extension is obtained by adding all roots of a cubic polynomial which has exactly one real root).

REMARK: Due to Dirichlet theorem, $\mathcal{O}_K^*$ is isomorphic to $\mathbb{Z}\times\{\pm 1\}$. Let $\mathcal{O}_K^{*,+} := \sigma^{-1}(\mathbb{R}^{>0})\cap\mathcal{O}_K^*$. Then **the group $\mathcal{O}_K^{*,+}$ is isomorphic to $\mathbb{Z}$.**

Consider the action of $\mathcal{O}_K^+\cong\mathbb{Z}^3$ on $\mathbb{R}^3=\mathbb{C}\times\mathbb{R}$

$$\rho^+(x)(z,t) := (z+\tau(x), t+\sigma(x)).$$

Let Γ be a semidirect product $\mathcal{O}_K^+\rtimes\mathcal{O}_K^{*,+}$, defined from the natural action of $\mathcal{O}_K^{*,+}$ on $\mathcal{O}_K^+$. **Define an action of Γ on $\mathbb{C}\times H$, where H is an upper halfplane**, as follows.

The subgroup $\mathcal{O}_K^+\subset\Gamma$ acts on $\mathbb{C}\times H=\mathbb{C}\times\mathbb{R}\times\mathbb{R}^{>0}$ by translations as above (trivially on the last argument), and $\mathcal{O}_K^{*,+}$ acts multiplicatively as

$$\rho^*(\xi)(z,z') := (\tau(\xi)z, \sigma(\xi)z').$$

Inue surfaces of type S^0

DEFINITION: The **Inoue surface of type S^0** is a quotient $(\mathbb{C} \times H)/\Gamma$.

Its properties: 1. It is a compact, complex solvmanifold

2. Inoue surface **admits a flat connection preserving the complex structure** (by construction).

3. Its cohomology are the same as of $S^3 \times S^1$.

4. The Inoue surface $M := (\mathbb{C} \times H)/\Gamma$ **has no complex curves**

Oeljeklaus-Toma manifolds

Let K be a number field which has $2t$ complex embedding denoted $\tau_i, \overline{\tau}_i$ and s real ones denoted σ_i, $s > 0$, $t > 0$.

Let $\mathcal{O}_K^{*,+} := \mathcal{O}_K^* \cap \bigcap_i \sigma_i^{-1}(\mathbb{R}^{>0})$. Choose in $\mathcal{O}_K^{*,+}$ a free abelian subgroup $\mathcal{O}_K^{*,U}$ of rank s such that the quotient $\mathbb{R}^s/\mathcal{O}_K^{*,U}$ is compact, where $\mathcal{O}_K^{*,U}$ is mapped to $\mathbb{R}^t$ as $\xi \longrightarrow \Big(\log(\sigma_1(\xi)), ..., \log(\sigma_t(\xi))\Big)$. Let $\Gamma := \mathcal{O}_K^+ \rtimes \mathcal{O}_K^{*,U}$.

DEFINITION: **An Oeljeklaus-Toma manifold** is a quotient $\mathbb{C}^t \times H^s/\Gamma$, where $\mathcal{O}_K^+$ acts on $\mathbb{C}^t \times H^t$ as

$$\zeta(x_1, ..., x_t, y_1, ..., y_s) = \Big(x_1 + \tau_1(\zeta), ..., x_t + \tau_t(\zeta), y_1 + \sigma_1(\zeta), ..., y_s + \sigma_s(\zeta)\Big),$$

and $\mathcal{O}_K^{*,U}$ as

$$\xi(x_1, ..., x_t, y_1, ..., y_s) = \Big(x_1, ..., x_t, \sigma_1(\xi)y_1, ..., \sigma_t(\xi)y_t\Big)$$

THEOREM: (Oeljeklaus-Toma) The OT-manifold $M := \mathbb{C}^t \times H^s/\Gamma$ **is a compact complex manifold,** without any non-constant meromorphic functions. When $s = 1, t = 1$, it is an Inoue surface of class S^0.

Complex geometry of Oeljeklaus-Toma manifolds

THEOREM: (Ornea-V.) Let K be a number field which has s real embeddings and $2t$ complex ones, $t = 1$, $s > 0$. **Then the corresponding Oeljeklaus-Toma manifold has no non-trivial complex subvarieties**.

THEOREM: (Sima Verbitsky) **OT-manifold has no curves.**

THEOREM: (Sima Verbitsky) Any complex surfaces which can be immersed to an OT-manifold **is isomorphic to an Inoue surface.**

THEOREM: (Battisti-Oeljeklaus) **OT-manifold has no divisors which are cohomologous to 0.**

Today's theorem is a work in progress with Liviu Ornea and Victor Vuletescu.

THEOREM: (Liviu Ornea, V., Victor Vuletescu)
Let $X \subset M$ be a complex subvariety in an OT-manifold. **Then X is flat affine** (that is, geodesically complete with respect to the flat affine connection on M). Moreover, X **is "generalized OT"** in the sense defined below.

Cousin groups

REMARK: Cousin groups were studied in 1960-ies by Akihiko Morimoto, who named them "(H.C) groups", and Klaus Kopfermann, who called them "toroidal". The term "Cousin group" most likely originated in the paper of Huckleberry and Margulis (1983).

DEFINITION: A complex, connected Lie group G is called **Cousin group** if all holomorphic functions on G are constant.

REMARK: Cousin groups are abelian. Indeed, suppose that G is a non-abelian, connected Lie group. Then its adjoint action on its Lie algebra $\mathfrak{g}$ is non-trivial. However, **this action defines a holomorphic functions from G to $\operatorname{End}(\mathfrak{g})$.**

This implies the following observation.

COROLLARY: A Cousin group is a quotient of $\mathbb{C}^n$ by a discrete subgroup.

Structure theorem for Cousin groups

A stronger result can be proven.

THEOREM: Let G be a Cousin group. **Then G is isomorphic to $(\mathbb{C}^*)^n/\Gamma$,** for some discrete subgroup Γ.

THEOREM: A quotient $(\mathbb{C}^*)^n/\Gamma$ **is a Cousin group if and only if Γ is Zariski dense in $(\mathbb{C}^*)^n$.**

Proof: Akhiezer, Dmitri N. *Lie group actions in complex analysis*, Aspects of Mathematics, E27. Friedr. Vieweg and Sohn, Braunschweig, 1995.

Cousin groups and Hodge structures

DEFINITION: Let $V_{\mathbb{R}}$ be a real vector space. **A (real) Hodge structure of weight** w on a vector space $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is a decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q}$, satisfying $\overline{V^{p,q}} = V^{q,p}$. It is called **integer Hodge structure** if one fixes an integer lattice $V_{\mathbb{Q}}$ or $V_{\mathbb{Z}}$ such that $V_{\mathbb{R}} = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ or $V_{\mathbb{R}} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$. A Hodge structure is equipped with $U(1)$-action, with $u \in U(1)$ acting as u^{p-q} on $V^{p,q}$. **Morphism** of integer Hodge structures is a map which is $U(1)$-invariant and preserves the lattice.

PROPOSITION: Let $V = (V_{\mathbb{Z}}, V_{\mathbb{C}} = \bigoplus_{p+q=w} V^{p,q})$ be an integer Hodge structure of even weight. Denote by Ψ the projection of $V_{\mathbb{C}}$ to $\bigoplus_{p\leqslant q} V^{p,q}$ along $\bigoplus_{p<q} V^{p,q}$, and let $W_{\mathbb{Z}} := \Psi(V_{\mathbb{Z}})$. Assume that V has no Hodge structure subquotients of weight 0. **Then $\bigoplus_{p\leqslant q} V^{p,q}/W_{\mathbb{Z}}$ is a Cousin group.**

Number fields and Hodge structures

EXAMPLE: Let K be a number field, $n = s + r$, where $2s$ is the number of complex embeddings of K and $r > 0$ the number of real ones. We are going to equip $V_{\mathbb C} := K \otimes_{\mathbb Q} \mathbb C$ with a Hodge structure as follows.

In each pair of complex conjugate complex embeddings of K to $\mathbb C$ choose one. Let $V^{2,0} \subset K \otimes_{\mathbb Q} \mathbb C$ be the direct sum of the chosen $\mathbb C$-components of $K \otimes_{\mathbb Q} \mathbb C$, and $V^{1,1}$ be the direct sum of all summands of V corresponding to real embeddings. Then $V_{\mathbb Z} := \mathcal O_K, V_{\mathbb C} := V^{0,2} \oplus V^{1,1} \oplus V^{0,2}$ is an integer Hodge structure on V. The corresponding Cousin group $\bigoplus_{p\leqslant q} V^{p,q}/V_{\mathbb Z}$ is called **the Cousin precursor of the OT-manifold**. OT-manifold is obtained as its $\mathbb Z^r$-quotient.

REMARK: **Hodge structures are not enough!** Any exact sequence of abelian varieties $0 \to A \to B \to C \to 0$, splits (after taking a finite covering). However, not any exact sequence of compact complex tori splits. Similarly, not any exact sequence of Cousin groups splits (even after taking the finite covers).

We want subgroups of our Cousing groups to be also Cousin. For this, splitting of exact sequences is necessary! This would not work without additional assumptions. **We need a Cousin analogue of "polarized Abelian variety".**

Polarized Hodge structures

DEFINITION: Polarization on a Hodge structrure of weight w is a $U(1)$-invariant non-degenerate 2-form $h \in V_{\mathbb{Q}}^* \otimes V_{\mathbb{Q}}^*$ (symmetric or antisymmetric depending on parity of w) which satisfies

$$-(\sqrt{-1}\,)^{p-q} h(x, \overline{x}) > 0 \qquad (1)$$

(**"Riemann-Hodge relations"**) for each non-zero $x \in V^{p,q}$.

PROPOSITION: Let $V_{\mathbb{Z}} := \mathcal{O}_K, V := V^{0,2} \oplus V^{1,1} \oplus V^{0,2}$ be the Hodge structure constructed above. **Then it is polarized.**

Proof: Consider $V = K \otimes_{\mathbb{Q}} \mathbb{C}$ as an algebra and let $h(x, y) := \operatorname{Tr}(L_x L_y)$, where $L_x(a) = xa$. ∎

DEFINITION: Let G be a Cousin group obtained from a Hodge structure as above. Then G is called **a Hodge-Cousin group**, and its subgroup obtained from a Hodge substructure – **a Hodge-Cousin subgroup**. A Hodge-Cousin group is called **polarized** if the corresponding Hodge structure is polarized.

THEOREM: Category of polarized Hodge-Cousin groups is abelian.

Proof: Easy. ∎

Generalized OT-manifolds

DEFINITION: Let G be a polarized Hodge-Cousin group, $G = V/\Gamma$, with $V = \mathbb{C}^n$ and $A \cong \mathbb{Z}^r$ a free abelian group acting on G by automorphisms. Assume that V is decomposed as $V = \mathbb{C}^s \times \mathbb{C}^r$, and A preserves the subset $\mathbb{C}^s \times \mathbb{H}^r$, where $\mathbb{H}^r$ is the product of upper half-planes, and acts on each component $\mathbb{H}$ by a (real) homothety, in such a way that the quotient $\mathbb{C}^s \times \mathbb{H}^r/(\Gamma \rtimes A)$ is a smooth, compact complex manifold. Assume, moreover, that G contains no non-trivial compact complex tori. The quotient $\mathbb{C}^s \times \mathbb{H}^r/(\Gamma \rtimes A)$ is called **generalized OT-manifold** if the following additional condition is satisfied.

Consider the linearized action of A on V, $\rho : A \longrightarrow \operatorname{End}(V)$. Since A is commutative, V is decomposed as $V \cong \bigoplus_i V_i$, with all V_i be A-invariant and 1-dimensional. Then for any non-zero element $a \in A$, the action of $\rho(a)$ on the product of first s components $\mathbb{C}^s \subset V$ has no real eigenvalues. **(O)**

REMARK: This bizarre set of assumptions is necessary, because the proof of flatness of subvarieties $Z \subset M$ is done by induction on $\dim Z$, and **we need the "generalized OT"-property to be inherited by all subvarieties.**

Generalized OT-manifolds (2)

REMARK: Clearly, an OT-manifold associated with a number field K **is a special case of a generalized OT-manifold.** In this case, the decomposition $V = \bigoplus_i V_i$ is identical to the standard splitting $V = \mathbb{C}^n$, with each component C corresponding to an embedding $\sigma : K \longrightarrow \mathbb{C}$. The corresponding action of $a \in A = \mathcal{O}_K^{*,u}$ is equal to a multiplication by $\sigma(a)$, and this is why the condition (O) holds.

THEOREM: (OVV: the main result of today's talk)
A complex subvariety of a generalized OT-manifold is also a generalized OT-manifold.

REMARK: It is not hard to see that **a closed Lie subgroup of a Hodge-Cousin subgroup is again a Hodge-Cousin group.** Therefore, it suffices to show that $Z \subset M$ is flat and hence corresponds to a Cousin subgroup.

Structure of tangent bundle for OT-manifolds

CLAIM: Let M be a generalized OT-manifold, and ∇ its flat affine connection. **Then TM is a direct sum of flat line bundles.**

Proof: The monodromy group of $M = \mathbb{C}^s \times \mathbb{H}^r/\Gamma$ is an image of $\Gamma = \mathcal{O}_K^+ \rtimes A$. However, $\mathcal{O}_K$ acts by parallel transport, hence it does not contribute to monodromy. Therefore, monodromy group of TM is identified with A. Since it is abelian, its action on T_xM is semisimple, and T_xM splits to a direct sum of monodromy-invariant line bundles. Using holonomy of the flat connection, we obtain that this decomposition is extended to a parallel decomposition of TM to a direct sum of flat line bunles. ∎

Subvarieties in flat manifolds

DEFINITION: Recall that **the algebraic dimension** of a complex manifold, denoted by $a(M)$, is the transcendental dimension of its field of global meromorphic functions.

Proposition 1: Let (M, ∇) be a complex flat affine manifold with (TM, ∇) decomposed as a direct sum of flat line bundles, and $Z \subset M$ an irreducible complex subvariety with $a(Z) = 0$. **Then Z is a flat affine submanifold of M.** Moreover, $TZ \subset TM|_Z$ is also a direct sum of line bundles.

Proof. Step 1: Let $k = \dim_{\mathbb{C}} Z$. Consider the line bundle $K^*Z \subset \Lambda^k TZ \subset \Lambda^k TM|_Z$. To prove that TZ is preserved by the connection ∇, **it suffices to show that K^*Z is preserved by ∇.**

Step 2: Since Z satisfies $a(Z) = 0$, **any non-degenerate morphism between line bundles on Z is an isomorphism**.

Step 3: Since $\Lambda^k TM|_Z$ is a direct sum of line bundles, the map $\Lambda^k TZ \subset \Lambda^k TM|_Z$ is an isomorphism of $\Lambda^k TZ$ and the direct sum of some line bunlde components of $\Lambda^k TM|_Z$, hence **it is preserved by ∇.** ∎

Curves in generalized OT-manifolds

PROPOSITION: Let M be a generalized OT-manifold. **Then M contains no non-trivial subvarieties of dimension 1.**

Proof. Step 1: Let $\tilde{M} = \mathbb{C}^s \times \mathbb{H}^r$ be the universal cover of M, $z_1, ..., z_s$ be the coordinates corresponding to $\mathbb{C}$ components and $z_{s+1}, ..., z_{r+s}$ the coordinates on $\mathbb{H}$ components. The 1-form $d\left(\sum_{i=s+1}^{s+r} \log(\operatorname{Im}(z_i))\right)$ is clearly A-invariant, and **the corresponding (1,1)-form $\omega_0 := dd^c\left(\sum_{i=s+1}^{s+r} \log(\operatorname{Im}(z_i))\right)$ is semi-positive.**

Step 2: Let $C \subset M$ be a 1-dimensional complex subvariety. Since ω_0 is exact, one has $\int_C \omega_0 = 0$, hence **C is tangent to the null-foliation Σ of ω_0.**

Step 3: Clearly, Σ is the foliation tangent to the $\mathbb{C}^s$-component of $\tilde{M} = \mathbb{C}^s \times \mathbb{H}^r$. Since A acts faithfully on leaves of Σ, it exchanges copies of C. Therefore, C can be lifted to the A-covering of M, which is a Cousin group, denoted as G. The Albanese map on G is identity, hence $\operatorname{Alb}(C)$ is a complex torus in G, which is impossible, because G contains no compact complex tori. This gives a contradiction. ■

Subvarieties in generalized OT-manifolds

Lemma 1: All generalized OT-subvarieties in a generalized OT-manifold are rigid (have no complex deformations).

Proof: See below.

THEOREM: Let $Z \subset M$ be an irreducible complex subvariety of a generalized OT-manifold. **Then Z is a flat subvariety of M.**

Proof. Step 1: To obtain flatness, it would suffice to prove that $a(Z) = 0$ (Proposition 1).

Step 2: Let d be the smallest dimension for which there exists $Z \subset M$ of positive algebraic dimension. Since M has no curves, we have $d > 1$. Then all proper subvarieties of a d-dimensional subvariety $Z \subset M$ are generalized OT-submanifolds of M. By Lemma 1, they are rigid. However, if $a(Z) > 0$, there exists a non-constant meromorphic function f on Z. Then the zero divisor of the function $f - \lambda$, for $\lambda \in \mathbb{C}$, is not rigid, which brings a contradiction. ∎

Rigidity of generalized OT-subvarieties

Lemma 1: **All generalized OT-subvarieties in a generalized OT-manifold are rigid** (have no complex deformations).

Proof: As shown above, $TM = TZ \oplus NZ$, with NZ a direct sum of line bundles: $NZ = \bigoplus L_i$. The condition "O" of the definition of generalized OT-manifold implies that all L_i are non-trivial. **To prove Lemma 1, it remains to show that $H^0(L_i) = 0$.** This is implied by the following lemma.

LEMMA: Let $M = P/A$ be a generalized OT-manifold, where $P \subset G$ is a convex subset of a Cousin group and $A \cong \mathbb{Z}^n$. Consider a non-trivial flat line bundle L on M with monodromy factorized through A, considered as a quotient group of $\pi_1(M)$. **Then L has no non-trivial holomorphic sections.**

Proof: By Battisti-Oeljeklaus, P has no global holomorphic functions. However, any section of L_i gives a rise to a holomorphic function on P which is (non-trivially) automorphic under the action of A, hence non-constant. ■

HAPPY BIRTHDAY!