Holography principle for twistor spaces

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Estruturas geométricas em variedades

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Calabi-Yau theorem

Definition 1: A **hyperkähler structure** on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1}\,\omega_K$ is a holomorphic symplectic form on (M, I).

Definition 2: A **hyperkähler manifold** is a complex, Kähler, holomorphically symplectic manifold.

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold **admits a unique hyperkähler metric in any Kähler class.**

REMARK: This means that **Definition 1 is equivalent to Definition 2**, but only for compact manifold.

REMARK: Twistors are used **to give a complex-analytic definition which works for non-compact and/or singular varieties.**

Twistor spaces for hyperkähler manifolds

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if M is compact, for generic a, b, c, (M, L) has no divisors (Fujiki).

DEFINITION: A **twistor space** $\mathrm{Tw}(M)$ of a hyperkähler manifold is **a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$.** More formally:

Let $\mathrm{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\mathrm{Tw}} = I_m \oplus I_J : T_x \mathrm{Tw}(M) \to T_x \mathrm{Tw}(M)$ satisfies $I_{\mathrm{Tw}}^2 = -\mathrm{Id}$. **It defines an almost complex structure on $\mathrm{Tw}(M)$.** This almost complex structure is known to be integrable (Obata, Salamon)

EXAMPLE: If $M = \mathbb{H}^n$, $\mathrm{Tw}(M) = \mathrm{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$

REMARK: For M compact, $\mathrm{Tw}(M)$ never admits a Kähler structure.

Twistor spaces for quaternionic-Kähler manifolds

DEFINITION: A **almost quaternionic-Kähler manifold** is a Riemannian manifold M equipped with a rank 3 sub-bundle $E \subset \mathfrak{so}(TM) \subset \operatorname{End}(M)$ which is pointwise isomorphic to $\mathfrak{su}(2)$ acting on TM as on a direct sum of $\frac{\dim_{\mathbb{R}} M}{4}$ of its fundamental representations (which have real dimension 4). This manifold is called **quaternionic-Kähler** if $\nabla(E) \subset E \otimes \Lambda^1 M$, where $\nabla: \operatorname{End}(M) \longrightarrow \operatorname{End}(M) \otimes \Lambda^1 M$ is the Levi-Civita connection.

DEFINITION: A **twistor space** $\operatorname{Tw}(M)$ of a quaternionic-Kähler manifold (M, g, E) is a total space of a unit sphere bundle on E, equipped with a complex structure $I_{\operatorname{Tw}} = I_m \oplus I_J : T_x \operatorname{Tw}(M) \to T_x \operatorname{Tw}(M)$ as in the hyperkähler case.

EXAMPLE: If $M = \mathbb{H}P^n$, then $\operatorname{Tw}(M) = \mathbb{C}P^{2n+1}$. In particular, $\operatorname{Tw}(S^4) = \mathbb{C}P^3$.

REMARK: Consider a compact quaternionic-Kähler manifold (M, g) with $\operatorname{Ric}(M) = \lambda g$, $\lambda > 0$. Then $\operatorname{Tw}(M)$ **is a holomorphically contact Fano manifold.** Conversely, **any Kähler-Einstein holomorphically contact Fano manifold is a twistor space of a compact quaternionic-Kähler manifold (M, g) with $\operatorname{Ric}(M) = \lambda g$, $\lambda > 0$.**

One can say that **hyperkähler geometry is holomorphic symplectic geometry, and quaternionic-Kähler is holomorphic contact geometry**

Twistor spaces for 4-dimensional Riemannian manifolds

DEFINITION: Let M be a Riemannian 4-manifold. Consider the action of the Hodge $*$-operator: $*: \Lambda^2 M \longrightarrow \Lambda^2 M$. Since $*^2 = 1$, the eigenvalues are ± 1, and one has a decomposition $\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$ onto **autodual** $(*\eta = \eta)$ and **anti-autodual** $(*\eta = -\eta)$ forms.

REMARK: If one changes the orientation of M, leaving metric the same, $\Lambda^+ M$ and $\Lambda^- M$ are exchanged. **Therefore,** $\dim \Lambda^2 M = 6$ **implies** $\dim \Lambda^\pm(M) = 3$**.**

REMARK: Using the isomorphism $\Lambda^2 M = \mathfrak{so}(TM)$, we interpret $\eta \in \Lambda^2_m M$ as an endomorphisms of $T_m M$. **Then the unit vectors $\eta \in \Lambda^+_m M$ correspond to oriented, orthogonal complex structures on $T_m M$.**

DEFINITION: Let $\operatorname{Tw}(M) := S\Lambda^+ M$ be the set of unit vectors in $\Lambda^+ M$. At each point $(m,s) \in \operatorname{Tw}(M)$, consider the decoposition $T_{m,s}\operatorname{Tw}(M) = T_m M \oplus T_s S\Lambda^+_m M$, induced by the Levi-Civita connection. Let I_s be the complex structure on $T_m M$ induced by s, $I_{S\Lambda^+_m M}$ the complex structure on $S\Lambda^+_m M = S^2$ induced by the metrics and orientation, and $\mathcal{I}: T_{m,s}\operatorname{Tw}(M) \longrightarrow T_{m,s}\operatorname{Tw}(M)$ be equal to $\mathcal{I}_s \oplus I_{S\Lambda^+_m M}$. An almost complex manifold $(\operatorname{Tw}(M), \mathcal{I})$ is called **the twistor space** of M.

Properties of twistor spaces for 4-dimensional manifolds

1. The almost complex structure $\mathcal{I}$ on $\operatorname{Tw}(M)$ **is a conformal invariant** of M. Moreover, **one can reconstruct the conformal structure** on M from the almost complex structure on $\operatorname{Tw}(M)$ and its anticomplex involution $(m, s) \longrightarrow (m, -s)$.

2. **$\operatorname{Tw}(M)$ is a complex manifold if and only if $W^+ = 0$,** where W^+ ("self-dual conformal curvature") is an autodual component of the curvature tensor. Such manifolds are called **conformally semi-flat** or **ASD (anti-selfdual).**

3. For a hyperkähler manifold of real dimension 4, the two definitions of twistor spaces coincide.

4. Let M be a compact ASD manifold. **Then $\operatorname{Tw}(M)$ does not admit a Kähler structure,** unless M is S^4 or $\mathbb{C}P^2$ (Hitchin). However, M **is often Moishezon.**

QUESTION: The twistor spaces are rationally connected. What else can one say about algebraic geometry of the Moishezon twistor spaces?

CONJECTURE: They are all rational.

Rational curves on $\operatorname{Tw}(M)$.

DEFINITION: An ample rational curve on a complex manifold M is a smooth curve $S \cong \mathbb{C}P^1 \subset M$ such that $NS = \bigoplus_{k=1}^{n-1} \mathcal{O}(i_k)$, with $i_k > 0$. It is called **a quasiline** if all $i_k = 1$.

THEOREM: ("twistor spaces are rationally connected")
Let M be a compact complex manifold containing a an ample rational line. **red any N points $z_1, ..., z_N$ can be connected by an ample rational curve.**

CLAIM: Let M be a hyperkähler manifold, $\operatorname{Tw}(M) \stackrel{\sigma}{\longrightarrow} M$ its twistor space, $m \in M$ a point, and $S_m = \mathbb{C}P^1 \times \{m\}$ the corresponding rational curve in $\operatorname{Tw}(M)$. **Then S_m is a quasiline.**

Proof: Since the claim is essentially infinitesimal, it suffices to check it when M is flat. **Then $\operatorname{Tw}(M) = \operatorname{Tot}(\mathcal{O}(1)^{\oplus 2p}) \cong \mathbb{C}P^{2p+1} \backslash \mathbb{C}P^{2p-1}$, and S_m is a section of $\mathcal{O}(1)^{\oplus 2p}$.** ■

Hypercomplex structures and twistor sections

DEFINITION: A **hypercomplex structure** on a manifold M is a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$.

REMARK: Twistor spaces for hypercomplex manifolds **are defined in the same way as for hyperkähler.**

REMARK: The twistor space **has many rational curves**. In fact, it is **rationally connected** (Campana).

DEFINITION: Denote by $\operatorname{Sec}(M)$ **the space of holomorphic sections** of the twistor fibration $\operatorname{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$.

DEFINITION: For each point $m \in M$, **the horizontal section of** π is a rational curve $C_m := \{m\} \times \mathbb{C}P^1$. The space of horizontal sections is denoted $\operatorname{Sec}_{hor}(M) \subset \operatorname{Sec}(M)$

REMARK: The space of horizontal sections of π is identified with M. The normal bundle $NC_m = \mathcal{O}(1)^{\dim M}$. Therefore, **some neighbourhood of** $\operatorname{Sec}_{hor}(M) \subset \operatorname{Sec}(M)$ **is a smooth manifold of dimension** $2\dim M$**.**

Hypercomplex structures defined in terms of quasilines

DEFINITION: Twistor data is the following collection of structures.

(i) A complex analytic variety Tw, equipped with a morphism $\pi : \text{Tw} \longrightarrow \mathbb{C}P^1$.

(ii) An anticomplex involution $\iota : \text{Tw} \longrightarrow \text{Tw}$ such that $\iota \circ \pi = \pi \circ \iota_0$.

(iii) A choice of connected component Hor of $\text{Sec}^\iota \subset \text{Sec}$.

Theorem 1: (Hitchin-Karlhede-Lindström-Roček)
Twistor data define a hypercomplex structure on $M =$ Hor if all curves in Hor are quasilines.

REMARK: Define a hypercomplex variety as a normal real analytic variety equipped with three complex structures I, J, K satisfying the quaternionic relations. **Then it is defined by the twistor data, and, conversely, Theorem 1 is true in this setting**.

Explicit formulas for hyperkähler structures

QUESTION: Let (M, I, J, K) be a hyperkähler manifold. **Define when (M, I, J, K) is "algebraic".**

REMARK: "Algebraic" in this context means "can be written explicitly in polynomial terms", "non-algebraic" – "no explicit polynomial formula (even locally)".

QUESTION: Can an open ball in a K3 surface (or some other compact hyperkähler manifold) with its hyperkähler structure **be equivalent to an open ball in a manifold with explicitly known hyperkähler structure, such as a quiver space?**

Answer: No.

Twistor spaces are non-Kähler

CLAIM: Suppose that (M, I, J, K) is a hyperkähler manifold such that (M, I) contains a compact, odd-dimensional complex subvariety Z. **Then** $\operatorname{Tw}(M)$ **is non-Kähler.**

Proof: Consider the fundamental class $[Z_I] \in H_*(M)$ of Z in $(M, I) \subset \operatorname{Tw}(M)$. Then $[Z_I] + [Z_{-I}] = 0$, giving $\int_{Z_I \cup Z_{-I}} \alpha = 0$ for each closed form $\alpha \in H^2(M)$. This is impossible if α is a Kähler form, because this integral is Riemannian volume of $Z_I \cup Z_{-I}$. ■

CLAIM: When M is compact and hyperkähler, $\operatorname{Tw}(M)$ never admits a Kähler structure.

Proof: Let ω be the standard Hermitian form of $\operatorname{Tw}(M)$. Then $dd^c\omega = \pi^*(\omega_{\mathbb{C}P^1}) \wedge \omega$, where $\omega_{\mathbb{C}P^1}$ is the Fubini-Study form on $\mathbb{C}P^1$ (Kaledin-V.). Therefore, $dd^c\omega$ **is a positive (2,2)-form**. For any Kähler form ω_0, **this would imply**

$$\int_{\operatorname{Tw}(M)} d\left(\omega_0^{\dim_{\mathbb{C}} M-1} \wedge d^c\omega\right) = \int_{\operatorname{Tw}(M)} \omega_0^{\dim_{\mathbb{C}} M-1} \wedge dd^c\omega > 0,$$

which is impossible by Stokes' theorem. ■

Holography principle

Theorem 1: (Holography principle for line bundles)
Let $S \subset M$ be an ample curve in a simply connected, connected complex manifold, which is covered by deformations of S. Consider a tubular neighbourhood $U \supset S$. Then, **for any holomorphic line bundle L on M, the space $H^0(U, L)$ is independent from the choice of U and S in its deformation class.**

REMARK: In these assumptions, $H^0(U, L)$ **is always finite-dimensional** (Hartshorne).

THEOREM: (Holography principle for meromorphic functions)
In assumptions of Theorem 1, **the space of meromorphic functions on U is equal to the space of meromorphic functions on M.**

DEFINITION: Given a complex manifold Z, denote by Mer(Z) the field of global meromorphic functions on Z, and let **the algebraic dimension** $a(Z)$ be the transcendence degree of Mer(Z).

REMARK: It could be infinite!

COROLLARY: $a(\mathrm{Tw}(M)) = a(U)$ **for any connected neighbourhood U of a quasiline** in a connected, simply connected M.

Domain of holomorphy

DEFINITION: Let M be a complex manifold. **Domain** over M is a holomorphic map $M' \longrightarrow M$ which is a local diffeomorphism.

DEFINITION: Let f be a holomorphic function on an open subset $U \subset M$. **Domain of holomorphy** for f is a domain $M_f \longrightarrow M$ such that f can be extended to a holomorphic function on M_f, but cannot be extended to any domain strictly containing M_f.

THEOREM: **Domain of holomorphy M_f exists and is unique for any function $f \in H^0(\mathcal{O}_U)$.**

Proof: Clearly, the etale space E of the sheaf of holomorphic functions on M is Hausdorff, and the projection of E to M is a local diffeomorphism. Then M_f is the connected component of E containing a germ of f. ■

DEFINITION: A complex manifold is **holomorphically convex** if it admits a proper holomorphic map to a Stein variety.

REMARK: By Cartan-Oka theorem, **a domain of holomorphy of f is always holomorphically convex if M is holomorphically convex.**

Moishezon twistor spaces

DEFINITION: A compact complex variety Z is called **Moishezon** if the ring of meromorphic functions on Z has algebraic dimension $\dim Z$, or, equivalently, if Z is bimeromorphic to a projective manifold.

CLAIM: Let M be a simply connected hyperkaähler manifold, and $\mathrm{Tw}(M)$ its twistor space. **Then $a(\mathrm{Tw}(M)) \leqslant \dim_{\mathbb{C}} \mathrm{Tw}(M)$.**

DEFINITION: A twistor space satisfying $a(\mathrm{Tw}(M)) = \dim_{\mathbb{C}} \mathrm{Tw}(M)$ is called **Moishezon**.

CLAIM: All Moishezon twistor spaces admit a rational, generically finite morphism to an open subset of a projective manifold.

THEOREM: Let V be a quaterionic Hermitian vector space, and $G \subset \mathrm{Sp}(V)$ a compact Lie group acting on V by quaternionic isometries. Denote by M the hyperkähler reduction of V. **Then $\mathrm{Tw}(M)$ is Moishezon.**

Local structure of hyperkähler manifolds

THEOREM: (Fujiki, 1987) Let M be a compact hyperkähler manifold, and $L \in S^2 \subset \mathbb{H}$ a generic induced complex structure. **Then M contains no divisors.**

COROLLARY: Let M be a compact hyperkähler manifold, and $D \subset \operatorname{Tw}(M)$ a divisor. **Then D is a union of several fibers of the projection $\pi : \operatorname{Tw}(M) \longrightarrow \mathbb{C}P^1$.**

Proof: Suppose that D contains a component which intersects generic fiber of π. By transversality, this intersection is a divisor, contradicting Fujiki's theorem. ■

REMARK: Algebraic dimension of $\operatorname{Tw}(M)$ is a local invariant of a hyperkähler manifold.

COROLLARY: Let M be a compact, simply connected hyperkähler manifold, M' a hyperkähler manifold obtained by hyperkähler reduction (such as Nakajima quiver variety), and $U \subset M$, $U' \subset M'$ open subsets. **Then U is never equivalent to U' as a hyperkähler manifold.**

Proof: $a(\operatorname{Tw}(U)) = a(\operatorname{Tw}(M)) = 1$, and
$a(\operatorname{Tw}(U')) = a(\operatorname{Tw}(M')) = \dim(\operatorname{Tw}(M'))$. ■

Complex manifolds and quasilines

REMARK: Let $S \subset M$ be a quasiline. Then, for an appropriate tubular neighbourhood $U \subset M$ of S, **"for every two points $x, y \in U$ close to S and far from each other, there is a unique deformation of S containing X and Y."**

More precisely:

CLAIM: Let $S \subset M$ be a quasi-line. Then, for any sufficiently small tubular neighbourhood $U \subset M$ of S, there exists a smaller tubular neighbourhood $W \subset U$, satisfying the following condition. Let Δ_S be the image of the diagonal embedding $\Delta_S : S \longrightarrow W \times W$. Then there exists an open neighbourhood V of Δ_S, properly contained in $W \times W$, such that **for any pair $(x, y) \in W \times W \backslash V$, there exists a unique deformation $S' \subset U$ of S containing x and y.**

COROLLARY: For any quasiline in M, its deformation space is 2(dim $M - 1$)-dimensional.

Proof of holography principle

THEOREM: (Holography Principle)
Let $S \subset M$ be a quasiline in a simply connected complex manifold, which is covered by deformations of S. Assume that M is equipped with a projection $\pi : M \longrightarrow S$ inducing identity on S. Consider a tubular neighbourhood $U \supset S$, and assume that $\pi : U \longrightarrow S$ has connected fibers. Then, **for any holomorphic line bundle L on M, the space $H^0(U, L)$ is independent from the choice of U and S in its deformation class.**

A (slighly) weaker statement.

THEOREM 1: Let $S \subset M$ be a quasiline, and L a holomorphic bundle on M. Assume that M is equipped with a projection $\pi : M \longrightarrow S$ inducing identity on S. Consider a sufficiently small tubular neighbourhood $U \supset S$, and a smaller tubular neighbourhood $V \subset U$. **Then the restriction map $H^0(U, L) \longrightarrow H^0(V, L)$ is an isomorphism.**

We deduce the "Holography principle" from Theorem 1.

Deducing the holography principle from Theorem 1

Step 1: Choose a continuous, connected family S_b of quasilines parametrized by B such that $\bigcup_{b\in B} S_b = M$. Find a tubular neighbourhood U_b for each S_b in such a way that an intersection $U_b \cap U_{b'}$ for sufficiently close b, b' always contains S_b and $S_{b'}$. **By Theorem 1, $H^0(U_b \cap U_{b'}, L) = H^0(U_b, L) = H^0(U_{b'}, L)$.**

Step 2: Since B is connected, **all the spaces $H^0(U_b, L)$ are isomorphic,** and these isomorphisms are compatible with the restrictions to the intersections $U_b \cap U_{b'}$.

Step 3: Let now $f \in H^0(U_b, L)$, and let $\tilde{M}_f$ be the **domain of holomorphy** for f, that is, a maximal domain (non-ramified over M) such that f admits a holomorphic extension to $\tilde{M}_f$. Since $\cup U_b = M$, and f can be holomorphically extended to any U_b, the domain $\tilde{M}_f$ is a covering of M. **Now, "holography principle" follows, because M is simply connected.** ■

Proof of Theorem 1

THEOREM 1: Let $S \subset M$ be a quasiline, and L a holomorphic bundle on M. Assume that M is equipped with a projection $\pi : M \longrightarrow S$ inducing identity on S. Consider a sufficiently small tubular neighbourhood $U \supset S$, and a smaller tubular neighbourhood $V \subset U$. **Then the restriction map $H^0(U, L) \longrightarrow H^0(V, L)$ is an isomorphism.**

Proof. Step 1: Fix a point $x_0 = \infty$ in $\mathbb{C}P^1$. A section of $L\big|_{\mathbb{C}P^1} = \mathcal{O}(d)$ is the same as meromorphic function having a pole of degree $\leqslant d$ at $\pi^{-1}(\infty)$.

Step 2: For each section $S_1 : \mathbb{C}P^1 \longrightarrow M$ of π, a degree d meromorphic function on S_1 is uniquely determined by its values at any $d+1$ points of S_1.

Step 3: Given a meromorphic function $f \in H^0(V, L)$ and a quasiline S_1 intersecting V in an open set, we can extend f to a meromorphic function $\tilde{f}$ on S_1 by computing its values at $d+1$ distinct points $z_1, ..., z_{d+1}$ of $S_1 \cap V$. Whenever S_1 is in V, this procedure gives $f\big|_{S_1}$. **By analytic continuation, the values of $\tilde{f}(z)$ at any $z \in S_1$ are independent from the choice of z_i and S_1.**

Step 4: Therefore, **$\tilde{f}$ is a well-defined meromorphic function on the union V_1 of all quasilines intersecting V.** For U sufficiently small, V_1 contains U. Therefore, any $f \in H^0(V, L)$ can be extended to U. ■

Hamiltonians

Let's define the hyperkähler reduction.

We denote the Lie derivative along a vector field as $\operatorname{Lie}_x : \Lambda^i M \longrightarrow \Lambda^i M$, and contraction with a vector field by $i_x : \Lambda^i M \longrightarrow \Lambda^{i-1} M$.

Cartan's formula: $d \circ i_x + i_x \circ d = \operatorname{Lie}_x$.

REMARK: Let (M, ω) be a symplectic manifold, G a Lie group acting on M by symplectomorphisms, and $\mathfrak{g}$ its Lie algebra. For any $g \in \mathfrak{g}$, denote by ρ_g the corresponding vector field. Then $\operatorname{Lie}_{\rho_g} \omega = 0$, giving $d(i_{\rho_g}(\omega)) = 0$. **We obtain that $i_{\rho_g}(\omega)$ is closed, for any $g \in \mathfrak{g}$.**

DEFINITION: A Hamiltonian of $g \in \mathfrak{g}$ is a function h on M such that $dh = i_{\rho_g}(\omega)$.

Moment maps

DEFINITION: (M, ω) be a symplectic manifold, G a Lie group acting on M by symplectomorphisms. **A moment map** μ of this action is a linear map $\mathfrak{g} \longrightarrow C^\infty M$ associating to each $g \in G$ its Hamiltonian.

REMARK: It is more convenient to consider μ as an element of $\mathfrak{g}^* \otimes_{\mathbb{R}} C^\infty M$, or (and this is most standard) **as a function with values in $\mathfrak{g}^*$**.

REMARK: Moment map **always exists** if M is simply connected.

DEFINITION: A moment map $M \longrightarrow \mathfrak{g}^*$ is called **equivariant** if it is equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$.

REMARK: $M \xrightarrow{\mu} \mathfrak{g}^*$ is a moment map iff for all $g \in \mathfrak{g}$, $\langle d\mu, g\rangle = i_{\rho_g}(\omega)$. Therefore, **a moment map is defined up to a constant $\mathfrak{g}^*$-valued function.** An equivariant moment map is is defined up to **a constant $\mathfrak{g}^*$-valued function which is G-invariant**.

DEFINITION: A G-invariant $c \in \mathfrak{g}^*$ is called **central**.

CLAIM: **An equivariant moment map exists whenever $H^1(G, \mathfrak{g}^*) = 0$.** In particular, when G is reductive and M is simply connected, an equivariant moment map exists.

Hyperkähler reduction

DEFINITION: Let G be a compact Lie group, ρ its action on a hyperkähler manifold M by hyperkähler isometries, and $\mathfrak{g}^*$ a dual space to its Lie algebra. **A hyperkähler moment map** is a G-equivariant smooth map $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ such that $\langle \mu_i(v), g\rangle = \omega_i(v, d\rho(g))$, for every $v \in TM$, $g \in \mathfrak{g}$ and $i = 1, 2, 3$, where ω_i is one three Kähler forms associated with the hyperkähler structure.

DEFINITION: Let ξ_1, ξ_2, ξ_3 be three G-invariant vectors in $\mathfrak{g}^*$. The quotient manifold $M /\!\!/\!\!/ G := \mu^{-1}(\xi_1, \xi_2, \xi_3)/G$ is called **the hyperkähler quotient** of M.

THEOREM: (Hitchin, Karlhede, Lindström, Roček)
The quotient $M /\!\!/\!\!/ G$ is hyperkaehler.

Holomorphic moment map

Let $\Omega := \omega_J + \sqrt{-1}\omega_K$. This is a holomorphic symplectic (2,0)-form on (M, I).

The proof of HKLR theorem. Step 1: Let μ_J, μ_K be the moment map associated with ω_J, ω_K, and $\mu_{\mathbb C} := \mu_J + \sqrt{-1}\,\mu_K$. Then $\langle d\mu_{\mathbb C}, g\rangle = i_{\rho_g}(\Omega)$ Therefore, $d\mu_{\mathbb C} \in \Lambda^{1,0}(M, I) \otimes \mathfrak{g}^*$.

Step 2: This implies that the map $\mu_{\mathbb C}$ is holomorphic. It is called **the holomorphic moment map.**

Step 3: By definition, $M /\!\!/\!\!/ G = \mu_{\mathbb C}^{-1}(c) /\!\!/ G$, where $c \in \mathfrak{g}^* \otimes_{\mathbb R} \mathbb C$ is a central element. **This is a Kähler manifold,** because it is a Kähler quotient of a Kähler manifold.

Step 4: We obtain 3 complex structures I, J, K on the hyperkähler quotient $M /\!\!/\!\!/ G$. **They are compatible in the usual way** (an easy exercise). ■

Twistor spaces and hyperkähler reduction

THEOREM: Let V be a quaterionic Hermitian vector space, and $G \subset \operatorname{Sp}(V)$ a compact Lie group acting on V by quaternionic isometries. Denote by M the hyperkähler reduction of V. **Then $\operatorname{Tw}(M)$ is Moishezon.**

Proof: $\operatorname{Tw}(M)$ is obtained as the space of stable $G_{\mathbb{C}}$-orbits in $\mu_{\mathbb{C}}^{-1}(0) \subset \operatorname{Tw}(V)$. The space $\operatorname{Tw}(V) = \mathbb{C}P^{2n+1} \backslash \mathbb{C}P^{2n-1}$ is algebraic. Averaging over G, we obtain that the field $G_{\mathbb{C}}$-invariant rational functions on $\operatorname{Tw}(M)$ has dimension $\dim \operatorname{Tw}(M)$, and $\operatorname{Tw}(M)$ is Moishezon. ■

COROLLARY: Let U be an open subset of a compact, simply connected hyperkähler manifold, and U' an open subset of a hyperkähler manifold obtained as $V /\!\!/\!\!/ G$, where V is flat and G reductive. **Then U is not isomorphic to U' as hyperkähler manifold**.

Proof: $\operatorname{Tw}(U')$ has many meromorphic functions, and $a(\operatorname{Tw}(U)) = 1$. ■