Non-hyperbolicity of hyperkähler manifolds

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VIKTOR KULIKOV’S SIXTIETH BIRTHDAY

Moscow, Steklov Institute, 7.12.2012
Plan of the talk

1. Introduce hyperkähler manifolds and their moduli. Define the birational moduli space as a quotient of a Teichmüller space $\mathbb{Per} = \text{SO}(b_2 - 3, 3)/\text{SO}(2) \times \text{SO}(b_2 - 3, 1)$ by an arithmetic group $\Gamma_I$.

2. Explore the non-Hausdorff properties of the birational moduli. Explain how the Moore's ergodic theorem is relevant.

3. Use the Brody's lemma to obtain non-hyperbolicity. Define twistor spaces and prove the Campana's theorem.

Holomorphically symplectic manifolds

**DEFINITION:** A hyperkähler structure on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**REMARK:** A hyperkähler manifold is holomorphically symplectic: $\omega_J + \sqrt{-1} \omega_K$ is a holomorphic symplectic form on $(M, I)$.

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

**DEFINITION:** For the rest of this talk, a hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.

**DEFINITION:** A compact hyperkähler manifold $M$ is called simple if $\pi_1(M) = 0$, $H^{2,0}(M) = \mathbb{C}$.

**Bogomolov's decomposition:** Any hyperkähler manifold admits a finite covering which is a product of a torus and several simple hyperkähler manifolds.

**Further on, all hyperkähler manifolds are assumed to be simple.**
Hilbert schemes

**THEOREM:** (a special case of Enriques-Kodaira classification) Let $M$ be a compact complex surface which is hyperkähler. Then $M$ is either a torus or a K3 surface.

**DEFINITION:** A Hilbert scheme $M^{[n]}$ of a complex surface $M$ is a classifying space of all ideal sheaves $I \subset \mathcal{O}_M$ for which the quotient $\mathcal{O}_M/I$ has dimension $n$ over $\mathbb{C}$.

**REMARK:** A Hilbert scheme is obtained as a resolution of singularities of the symmetric power $\text{Sym}^n M$.

**THEOREM:** (Beauville) A Hilbert scheme of a hyperkähler surface is hyperkähler.
EXAMPLES.

EXAMPLE: A Hilbert scheme of K3 is simple and hyperkähler.

EXAMPLE: Let $T$ be a torus. Then it acts on its Hilbert scheme freely and properly by translations. For $n = 2$, the quotient $T^{[n]}/T$ is a Kummer K3-surface. For $n > 2$, a universal covering of $T^{[n]}/T$ is called a generalized Kummer variety.

REMARK: There are 2 more “sporadic” examples of compact hyperkähler manifolds, constructed by K. O’Grady. All known simple hyperkaehler manifolds are these 2 and two series: Hilbert schemes of K3, and generalized Kummer.
**The Teichmüller space and the mapping class group**

**Definition:** Let $M$ be a compact complex manifold, and $\text{Diff}_0(M)$ a connected component of its diffeomorphism group (**the group of isotopies**). Denote by $\tilde{\text{Teich}}$ the space of complex structures on $M$, and let $\text{Teich} := \tilde{\text{Teich}}/\text{Diff}_0(M)$. We call it **the Teichmüller space**.

**Remark:** $\text{Teich}$ is a finite-dimensional complex space (Kodaira-Spencer-Kuranishi-Douady), but often non-Hausdorff.

**Definition:** Let $\text{Diff}_+(M)$ be the group of oriented diffeomorphisms of $M$. We call $\Gamma := \text{Diff}_+(M)/\text{Diff}_0(M)$ **the mapping class group**. The coarse moduli space of complex structures on $M$ is a connected component of $\text{Teich}/\Gamma$.

**Remark:** This terminology is standard for curves.

**REMARK:** For hyperkähler manifolds, it is convenient to take for $\text{Teich}$ the space of all complex structures of hyperkähler type, that is, holomorphically symplectic and Kähler. It is open in the usual Teichmüller space.
The Bogomolov-Beauville-Fujiki form

**THEOREM:** (Fujiki). Let $\eta \in H^2(M)$, and dim $M = 2n$, where $M$ is hyperkähler. Then $\int_M \eta^{2n} = c q(\eta, \eta)^n$, for some primitive integer quadratic form $q$ on $H^2(M, \mathbb{Z})$, and $c > 0$ an integer number.

**Definition:** This form is called **Bogomolov-Beauville-Fujiki form.** It is defined by the Fujiki’s relation uniquely, up to a sign. The sign is determined from the following formula (Bogomolov, Beauville)

$$
\lambda q(\eta, \eta) = \int_X \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} -
\frac{n-1}{n} \left( \int_X \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^n \right) \left( \int_X \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1} \right)
$$

where $\Omega$ is the holomorphic symplectic form, and $\lambda > 0$. 

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Computation of the mapping class group

**Theorem:** (Sullivan) Let $M$ be a compact, simply connected Kähler manifold, $\dim_{\mathbb{C}} M \geq 3$. Denote by $\Gamma_0$ the group of automorphisms of an algebra $H^*(M, \mathbb{Z})$ preserving the Pontryagin classes $p_i(M)$. Then the natural map $\text{Diff}_+(M)/\text{Diff}_0 \longrightarrow \Gamma_0$ has finite kernel, and its image has finite index in $\Gamma_0$.

**Theorem:** Let $M$ be a simple hyperkähler manifold, and $\Gamma_0$ as above. Then

(i) $\Gamma_0|_{H^2(M, \mathbb{Z})}$ is a finite index subgroup of $O(H^2(M, \mathbb{Z}), q)$.

(ii) The map $\Gamma_0 \longrightarrow O(H^2(M, \mathbb{Z}), q)$ has finite kernel.
The period map

Remark: For any $J \in \text{Teich}$, $(M, J)$ is also a simple hyperkähler manifold, hence $H^{2,0}(M, J)$ is one-dimensional.

Definition: Let $P : \text{Teich} \rightarrow \mathbb{P}H^2(M, \mathbb{C})$ map $J$ to a line $H^{2,0}(M, J) \in \mathbb{P}H^2(M, \mathbb{C})$. The map $P : \text{Teich} \rightarrow \mathbb{P}H^2(M, \mathbb{C})$ is called the period map.

Remark: $P$ maps Teich into an open subset of a quadric, defined by

$$\text{Per} := \{l \in \mathbb{P}H^2(M, \mathbb{C}) \mid q(l, l) = 0, q(l, \bar{l}) > 0\}.$$ 

It is called the period space of $M$.

Remark: $\text{Per} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$
**Birational Teichmüller moduli space**

**DEFINITION:** Let $M$ be a topological space. We say that $x, y \in M$ are **non-separable** (denoted by $x \sim y$) if for any open sets $V \ni x, U \ni y$, $U \cap V \neq \emptyset$.

**THEOREM:** (Huybrechts) Two points $I, I' \in \text{Teich}$ are non-separable if and only if there exists a bimeromorphism $(M, I) \rightarrow (M, I')$ which is non-singular in codimension 2.

**DEFINITION:** The space $\text{Teich}_b := \text{Teich} / \sim$ is called the **birational Teichmüller space** of $M$.

**THEOREM:** The period map $\text{Teich}_b \xrightarrow{\text{Per}} \text{Per}$ is an isomorphism, for each connected component of $\text{Teich}_b$.

**DEFINITION:** Let $M$ be a hyperkaehler manifold, $\text{Teich}_b$ its birational Teichmüller space, and $\Gamma$ the mapping class group. The quotient $\text{Teich}_b / \Gamma$ is called the **birational moduli space** of $M$. 
**Monodromy group and the birational moduli space**

**THEOREM:** Let $(M, I)$ be a hyperkähler manifold, and $W$ a connected component of its birational moduli space. Then $W$ is isomorphic to $\mathbb{P}_{\text{Per}}/\Gamma$, where $\mathbb{P}_{\text{Per}} = SO(b_2 - 3, 3)/SO(2) \times SO(b_2 - 3, 1)$ and $\Gamma$ is an arithmetic group in $O(H^2(M, \mathbb{R}), q)$, called the **monodromy group**.

**REMARK:** $\Gamma_I$ is a group generated by monodromy of the Gauss-Manin local system on $H^2(M)$.

**A CAUTION:** Usually “the global Torelli theorem” is understood as a theorem about Hodge structures. For K3 surfaces, the **Hodge structure on $H^2(M, \mathbb{Z})$ determines the complex structure**. For $\dim_{\mathbb{C}} M > 2$, it is false.

**REMARK:** Further on, **I shall freely identify $\mathbb{P}_{\text{Per}}$ and $\text{Teich}_b$**.
Ergodicity of the monodromy group action

The moduli space $\mathbb{P}e/\Gamma_I$ is extremely non-Hausdorff.

**THEOREM:** (Calvin C. Moore, 1966) Let $\Gamma$ be an arithmetic lattice in a non-compact simple Lie group $G$ with finite center, and $H \subset G$ a non-compact subgroup. Then the left action of $\Gamma$ on $G/H$ is **ergodic**, that is, for all $\Gamma$-invariant measurable subsets $Z \subset G/H$, either $Z$ has measure 0, or $G/H \setminus Z$ has measure 0.

**REMARK:** This implies that “almost all” $\Gamma$-orbits in $G/H$ are dense.

**THEOREM:** Let $\mathbb{P}e$ be a component of a birational Teichmüller space, and $\Gamma$ its monodromy group. Let $\mathbb{P}e_e$ be a set of all points $L \subset \mathbb{P}e$ such that the orbit $\Gamma \cdot L$ is dense. *Then* $Z := \mathbb{P}e \setminus \mathbb{P}e_e$ *has measure 0.*

**Proof. Step 1:** Let $G = SO(b_2 - 3, 3)$, $H = SO(2) \times SO(b_2 - 3, 1)$. *Then* $\Gamma$-action on $G/H$ is **ergodic**, by Moore’s theorem.

**Step 2:** Ergodic orbits are dense, non-ergodic orbits have measure 0. ■
Ergodic complex structures

**COROLLARY:** For each hyperkähler manifold $M$ there exists a complex structure $I$ such that any other complex structure $I'$ in the same deformation class can be obtained as a limit of $\varphi_i I$, where $\varphi_i$ is a sequence of isotopies.

**DEFINITION:** We call a complex structure $I$ **ergodic** if its orbit in $\mathbb{Per}$ is dense.

**PROBLEM:** Nobody has produced a concrete example of an ergodic complex structure (so far).
Kobayashi hyperbolic manifolds

**DEFINITION:** An entire curve is a non-constant map $\mathbb{C} \longrightarrow M$.

**DEFINITION:** A compact complex manifold $M$ is called Kobayashi hyperbolic, if there exist no entire curves $\mathbb{C} \longrightarrow M$.

**THEOREM:** (Brody, 1975)
Let $I_i$ be a sequence of complex structures on $M$ which are not hyperbolic, and $I$ its limit. Then $(M, I)$ is also not hyperbolic.

**CONJECTURE:** All hyperkähler manifolds are non-hyperbolic.

**REMARK:** This conjecture would follow if we produce an ergodic complex structure which is non-hyperbolic.
Twistor spaces and hyperkähler geometry

**DEFINITION:** A hyperkähler structure on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**DEFINITION:** Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1.\}$

**DEFINITION:** A twistor space $\text{Tw}(M)$ of a hyperkähler manifold is a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$. More formally:

Let $\text{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on $M$ induced by $J \in S^2 \subset \mathbb{H}$. Let $I_J$ denote the complex structure on $S^2 \cong \mathbb{C}P^1$.

The operator $I_{\text{Tw}} = I_m \oplus I_J : T_x \text{Tw}(M) \to T_x \text{Tw}(M)$ satisfies $I_{\text{Tw}}^2 = -\text{Id}$. It defines an almost complex structure on $\text{Tw}(M)$. This almost complex structure is known to be integrable (Obata).
Rational curves on $\text{Tw}(M)$.

**DEFINITION:** An ample rational curve on a complex manifold $M$ is a smooth curve $S \cong \mathbb{C}P^1 \subset M$ such that $NS = \bigoplus_{k=1}^{n-1} \mathcal{O}(i_k)$, with $i_k > 0$. It is called a quasiline if all $i_k = 1$.

**CLAIM:** Let $M$ be a compact complex manifold containing an ample rational line. Then any $N$ points $z_1, ..., z_N$ can be connected by an ample rational curve.

**CLAIM:** Let $M$ be a hyperkähler manifold, $\text{Tw}(M) \xrightarrow{\sigma} M$ its twistor space, $m \in M$ a point, and $S_m = \mathbb{C}P^1 \times \{m\}$ the corresponding rational curve in $\text{Tw}(M)$. Then $S_m$ is a quasiline.

**Proof:** Since the claim is essentially infinitesimal, it suffices to check it when $M$ is flat. Then $\text{Tw}(M) = \text{Tot}(\mathcal{O}(1)^{\oplus 2p}) \cong \mathbb{C}P^{2p+1} \setminus \mathbb{C}P^{2p-1}$, and $S_m$ is a section of $\mathcal{O}(1)^{\oplus 2p}$. ■
Non-hyperbolicity of hyperkähler manifolds

Entire curves in twistor fibers

**THEOREM: (F. Campana)**
Let $M$ be a hyperkähler manifold, and $\text{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$ its twistor projection. Then there exists an entire curve in some fiber of $\pi$.

**Proof:** The space of rational curves in $\text{Tw}(M)$ is not compact, because $M$ is not Moishezon. Take a sequence $s_i : \mathbb{C}P^1 \to \text{Tw}(M)$ of rational curves which does not converge. Then $\lim_i |ds_i(I_i)| = \infty$ for some sequence $I_i \in \mathbb{C}P^1$. Take a subsequence for which $I_i$ converges to some $I$. Then $\pi^{-1}(I)$ contains an entire curve obtained as a limit of $s_i$, by Brody’s theorem. ■

**COROLLARY:** Let $N \subset \mathbb{P}\text{er}$ be the set of all non-hyperbolic complex structures. Then $N$ contains a point on each rational curve $S \subset \mathbb{P}\text{er}$ obtained from a hyperkähler structure.

**COROLLARY:** $N$ has Hausdorff codimension $\leq 2$.

**REMARK:** Such rational curves $S$ correspond to 3-dimensional subspaces $W \subset H^2(M, \mathbb{R})$, with $\mathbb{P}\text{er} = \text{Gr}_{++}(H^2(M, \mathbb{R}))$, $S_W = \text{Gr}_{++}(W)$. To prove non-hyperbolicity it would suffice to show that the set of ergodic points contains $S_W$ for some $W \subset H^2(M, \mathbb{R})$. 

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Divisors in the moduli space

Instead of taking a $\Gamma$-orbit of a point $s \in \text{Per}$, let's take an orbit of a subvariety.

**DEFINITION:** Given non-zero $\eta \in H^2(M, \mathbb{R})$, denote by $\text{Per}_\eta \subset \text{Per}$ the set of all $I \in \text{Per}$ such that $\eta \in H^{1,1}(M, I)$.

**EXAMPLE:** When $q(\eta, \eta) > 0$, and $\eta$ is integer, $\eta$ or $-\eta$ is ample when $\text{Pic}(M, I) = \langle \eta \rangle$ (Huybrechts, Boucksom). The space $\text{Per}_\eta \subset \text{Per}$ is called the polarized Teichmüller space of $M$. It is a symmetric space. Its quotient $\text{Per}_\eta / \Gamma$ is quasiprojective, by Bailey-Borel's theorem, and Hausdorff.

**THEOREM:** (Anan' in-V.) For any integer $\eta$, the quotient $\text{Per}_\eta / \Gamma$ is dense in the corresponding moduli space $\text{Per} / \Gamma$.

For today's talk, we are interested in $q(\eta, \eta) = 0$. 

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Non-hyperbolicity of hyperkähler manifolds

Holomorphic Lagrangian fibrations

THEOREM: (Matsushita, 1997)
Let \( \pi : M \to X \) be a surjective holomorphic map from a hyperkähler manifold \( M \) to \( X \), with \( 0 < \dim X < \dim M \). Then \( \dim X = \frac{1}{2} \dim M \), and the fibers of \( \pi \) are holomorphic Lagrangian (this means that the symplectic form vanishes on \( \pi^{-1}(x) \)).

DEFINITION: Such a map is called holomorphic Lagrangian fibration.

REMARK: The base of \( \pi \) is conjectured to be rational. Hwang (2007) proved that \( X \cong \mathbb{C}P^n \), if it is smooth. Matsushita (2000) proved that it has the same rational cohomology as \( \mathbb{C}P^n \).

REMARK: The base of \( \pi \) has a natural flat connection on the smooth locus of \( \pi \). The combinatorics of this connection can be used to determine the topology of \( M \) (Strominger-Yau-Zaslow, Kontsevich-Soibelman).

REMARK: A manifold admitting a holomorphic Lagrangian fibration is non-hyperbolic, because it contains a torus.
The hyperkähler SYZ conjecture

**CONJECTURE:** (Tyurin, Bogomolov, Hassett-Tschinkel, Huybrechts, Sawon). Any hyperkähler manifold can be deformed to a manifold admitting a holomorphic Lagrangian fibration.

**A trivial observation:** Let $\pi : M \to X$ be a holomorphic Lagrangian fibration, and $\omega_X$ a Kähler class on $X$. Then $\eta := \pi^* \omega_X$ is nef, and satisfies $q(\eta, \eta) = 0$.

**The hyperkähler SYZ conjecture:** Let $L$ be a nef line bundle on a hyperkähler manifold, with $q(L, L) = 0$. Then $L$ is semiample. Here $q$ is the Bogomolov-Beauville form.

**Theorem:** (Kamenova-V.) Let $\eta \in H^2(M, \mathbb{Z})$ be a cohomology class satisfying $q(\eta, \eta) = 0$, and $I \in \text{Per}_\eta$ a complex structure for which $\eta$ is semiample. Then $\eta$ is semiample for a dense, open subset of $\text{Per}_\eta$. 
Non-hyperbolicity of hyperkähler manifolds

**COROLLARY:** (Kamenova-V.) Let $M$ be a hyperkähler manifold which has a deformation admitting a holomorphic Lagrangian fibration. Then $M$ is non-hyperbolic.

**Proof.** **Step 1:** Let $\eta$ be a nef class associated with a holomorphic Lagrangian fibration. Then $\eta$ is semiample for a dense, open subset $\text{Per}_{\eta}^{sa} \subset \text{Per}_{\eta}$. Since $\Gamma \cdot \text{Per}_{\eta}$ is dense in $\text{Per}$, $\Gamma \cdot \text{Per}_{\eta}^{sa}$ is also dense.

**Step 2:** All points of $\Gamma \cdot \text{Per}_{\eta}^{sa}$ are non-hyperbolic, and the set $N \supset \Gamma \cdot \text{Per}_{\eta}^{sa}$ of non-hyperbolic points is closed in $\text{Per}$. Therefore, $N = \text{Per}$. 

**EXAMPLE:** All known examples of hyperkähler manifolds (Hilbert schemes of K3, generalized Kummer, 2 of O’Grady’s examples) have a deformation admitting a Lagrangian fibration. Therefore, they are non-hyperbolic.