# Complex subvarieties in homogeneous complex manifolds

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## Homogeneous complex manifolds

**DEFINITION:** A complex manifold $M$ is called **homogeneous** if its automorphism group acts transitively.

**Examples of compact homogeneous manifolds:**

0. Flag spaces and partial flag spaces.

1. Calabi-Eckmann and Hopf manifolds.

2. Tori.

3. Let $G$ be a compact, even-dimensional Lie group. Then $G$ **admits a left-invariant complex structure** (H. Samelson, 1953).

## Hopf surface

**The (classical) Hopf surface**. Fix $\alpha \in \mathbb{C}$, $|\alpha| > 1$. Consider the quotient $H = (\mathbb{C}^2\backslash 0)/\langle\mathbb{Z}\rangle$, with $\mathbb{Z}$ acting on $\mathbb{C}^2$ by $(x, y) \longrightarrow (\alpha x, \alpha y)$. It is called **the Hopf surface**. Topologically the Hopf surface is isomorphic to $S^1 \times S^3$ (hence, non-Kähler). The elliptic curve $T^2 = \mathbb{C}^*/\langle\alpha\rangle$ acts on $H$ by $t, (x, y) \longrightarrow (tx, ty)$. This action is free, and its quotient is $\mathbb{C}P^1$. The Hopf surface is a **principal elliptic fibration.** Topologically, it's a product of a Hopf fibration $S^3 \longrightarrow S^2$ and a circle.

## Calabi-Eckmann manifolds

Fix $\alpha \in \mathbb{C}$, $\alpha$ non-real, $|\alpha| > 1$. Consider a subgroup

$$G := \{e^t \times e^{\alpha t} \subset \mathbb{C}^* \times \mathbb{C}^*, \quad t \in \mathbb{C}\} \subset \mathbb{C}^* \times \mathbb{C}^*$$

within $\mathbb{C}^* \times \mathbb{C}^*$. It is clearly co-compact and closed, with $\mathbb{C}^* \times \mathbb{C}^*/G$ being an elliptic curve $\mathbb{C}^*/\langle\alpha\rangle$.

Now, let $M := (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/G$, with $G \subset \mathbb{C}^* \times \mathbb{C}^*$ acting on $(\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)$ by $(t_1, t_2)(x, y) \longrightarrow (t_1 x, t_2 y)$. Clearly, $M$ is fibered over

$$\mathbb{C}P^{n-1} \times \mathbb{C}P^{m-1} = (\mathbb{C}^n \backslash 0) \otimes (\mathbb{C}^m \backslash 0)/\mathbb{C}^* \times \mathbb{C}^*$$

with a fiber $\mathbb{C}^* \times \mathbb{C}^*/G$, which is an elliptic curve. Then $M$ is called **the Calabi-Eckmann manifold**. It is diffeomorphic to $S^{2n-1} \times S^{2m-1}$. The group $U(n) \times U(m)$ acts on $M$ transitively.

**We obtained a homogeneous complex structure on $S^{2n-1} \times S^{2m-1}$.**

It is non-Kähler, because $H^2(M) = 0$.

## Principal toric fibrations

**DEFINITION: A complex principal toric fibration** $M$ is a complex manifold equipped with a free holomorphic action of a compact complex torus $T$.

**Such a manifold is fibered over $M/T$, with fiber $T$.**

It is a principal $T$-bundle: all fibers are identified with $T$, with $T$ acting on fibers freely.

**To trivialize a principal group bundle it means to find a section**.

## Borel-Remmert-Tits theorem

**Borel-Remmert-Tits theorem:** Let $M$ be a compact, complex, simply connected homogeneous manifold . Then $M$ is a principal toric fibration, with a base which is a homogeneous, rational projective manifold.

**Proof:** Let $K^{-1} = \Lambda_{\mathbb{C}}^{\dim M}(TM)$ be the anticanonical class of $M$. Since $TM$ is globally generated, the same is true for $K^{-1}$. This gives a $G$-invariant morphism

$$M \xrightarrow{\pi} \mathbb{P}H^0(K^{-1}).$$

**The fibers $F$ of $\pi$ are homogeneous with trivial canonical class, and its base is homogeneous and projective (hence, rational).** The fundamental group of $F$ is a quotient of $\pi_2(X)$, as follows from the long exact sequence of homotopy groups for a Serre's fibration:

$$\pi_2(X) \longrightarrow \pi_1(F) \longrightarrow \pi_1(M) = 0$$

Therefore, $\pi_1(F)$ is abelian. **It remains to show that it is a torus.**

## Homogeneous manifolds with trivial canonical class

**LEMMA:** Let $F$ be a compact, complex, homogeneous manifold with $\pi_1(F)$ abelian and a trivial anticanonical class $K^{-1}$. **Then $F$ is a torus.**

**Proof:** The sheaf of holomorphic vector fields on $M$ is globally generated. Taking a vector field $v_1$ and multiplying it by general vector fields $v_2, ... v_n$, we obtain a section of $K^{-1}$, which is non-zero for general $v_i$, and therefore non-degenerate. We obtain that $v_i$ are linearly independent everywhere. Taking the corresponding flows of diffeomorphisms, we obtain that $F$ is a quotient of a holomorphic Lie group $G$ by a cocompact lattice. **Since $\pi_1(F)$ is abelian, $G$ is commutative, and $T$ is a torus.** ■

## Positive elliptic fibrations

**DEFINITION:** Let $M \xrightarrow{\pi} X$ be an elliptic fibration, $M$ compact. We say that $M$ is a **positive elliptic fibration**, if for some Kähler class $\omega$ on $X$, $\pi^*\omega$ is exact. ("Kähler class" is a cohomology class of a Kähler form.)

**Examples:**

1. **Hopf manifold**, $H^2(M) = 0$, hence positive.
2. **Calabi-Eckmann manifold** (same).
3. $SU(3)$ is elliptically fibered over the flag manifold $F(2,3)$, also $H^2(M) = 0$.

## Subvarieties of positive elliptic fibrations

**Theorem:** Let $M \xrightarrow{\pi} X$ be a positive elliptic $T$ fibration, and $Z \subset M$ be a subvariety, of positive dimension $m$. **Then $Z$ is $T$-invariant.**

**Proof:** Let $\omega_0 = \pi^* \omega$ be a pullback of a Kähler form which is exact. Then

$$\int_Z \omega_0^m = 0.$$

On the other hand, all eigenvalues of $\omega_0 \Big|_Z$ are non-negative, and all are positive, unless $Z$ is tangent to the action of $T$. In a point where $Z$ is not tangent to $T$, the form $\omega_0^m$ is positive, and **in this case the integral $\int_Z \omega_0^m$ is also positive.**

## Positive toric fibrations

**DEFINITION:** Let $M \xrightarrow{\pi} X$ be a complex principal toric fibration, $M$ compact, with fiber $T$. We say that $\pi$ is **convex** if $\pi^*\omega$ is exact for some Kähler form $\omega$. We say that $\pi$ is **positive** if for any proper complex subtorus $T' \subset T$, the corresponding quotient fibration $M/T' \longrightarrow X$ is convex.

**EXAMPLE:** Let $M$ be a complex, compact homogeneous manifold with $H^2(M) = 0$ (e.g. a Lie group), and $M \xrightarrow{\pi} X$ the Borel-Remmert-Tits toric fibration. Assume that the fiber of $\pi$ have no proper subtori (easy to insure by taking a generic invariant complex structure). **Then $M$ is positive.**

## Subvarieties in principal toric fibrations

**THEOREM:** Consider an irreducible complex subvariety $Z \subset M$ of a positive principal toric fibration $M \xrightarrow{\pi} X$, with fiber $T$. **Then $Z$ is $T$-invariant, or is contained in a fiber of $\pi$.**

**Proof:** 1. For any positive-dimensional subvariety $Z_0 \subset X$, the restriction of $\pi$ to $Z_0$ **has no multisections** (because $\int_Z \omega_0^m$ must vanish).

2. Given a Kähler manifold $A$ with an action of $T$, consider an associated fiber bundle $M \times_T A$ over $X$. Unless $T$ acts on $A$ trivially, $M \times_T A$ is also convex, hence **admits no multisections.**

3. If $Z \subset M$ is not $T$-invariant, it provides us with a multisection from $X$ to $M \times_T A$, where $A$ is the space of deformations of the fiber $Z \cap \pi^{-1}(t_0)$. It is convex (step 2). **Cannot have multisections!** Contradiction. ■

## Open questions

1. **THEOREM:** Let $M \xrightarrow{\pi} X$, $\dim_{\mathbb{C}} X > 1$, be a positive elliptic $T$ fibration, and $F$ a stable reflexive sheaf on $M$. **Then $F \cong L \otimes \pi^* F_0$, where $L$ is a line bundle, and $F_0$ a stable coherent sheaf on $X$.**

**Is there a similar result for positive torus fibrations?**

2. Is it possible to define positivity for $\mathbb{C}^n$-fibrations? For $\mathbb{C}$-fibrations it is possible in terms of curvature; all subvarieties would be also $\mathbb{C}$-invariant.