3-webs and connections on the space of quasilines

Misha Verbitsky

Florida International University,
Aug. 24, 2012
Holomorphic 3-webs.

**DEFINITION:** Let $M$ be a complex manifold, and $S_1, S_2, S_3$ integrable, pairwise transversal holomorphic sub-bundles in $TM$, of dimension $\frac{1}{2}\dim M$. Then $(S_1, S_2, S_3)$ is called a **holomorphic 3-web** on $M$.

**REMARK:** On smooth manifolds, the theory of 3-webs is due to Chern and Blaschke (1930-ies).

**THEOREM:** (Ph. D. thesis of Chern, 1936) Let $S_1, S_2, S_3$ be a holomorphic 3-web on a complex manifold $M$. Then there exists a unique holomorphic connection $\nabla$ on $M$ which preserves the sub-bundles $S_i$, and such that its torsion $T$ satisfies $T(S_1, S_2) = 0$. 


Holomorphic $SL(2)$-webs.

**DEFINITION:** A holomorphic 3-web $S_1$, $S_2$, $S_3$ on a complex manifold $M$ is called an $SL(2)$-web if

- the projection operators $P_{i,j}$ of $TM$ to $S_i$ along $S_j$ generate the standard action of $\text{Mat}(2)$ on $\mathbb{C}^2 \otimes \mathbb{C}^n$,
- for any nilpotent $v \in \text{Mat}(2)$, the bundle $v(TM) \subset TM$ is involutive.

**REMARK:** The set of $v \in \text{Mat}(2)$ with $\text{rk} v = 1$ satisfies $\mathbb{P}V = \mathbb{C}P^1$, hence the sub-bundles $v(TM) \subset TM$ are parametrized by $\mathbb{C}P^1$. An $SL(2)$-web is determined by a set of sub-bundles $S_t \subset TM$, $t \in \mathbb{C}P^1$, which are pairwise transversal and involutive.

**THEOREM:** Let $S_t \subset TM$, $t \in \mathbb{C}P^1$ be an $SL(2)$-web on $M$, and $t_1, t_2, t_3 \in \mathbb{C}P^1$ distinct points. Then the Chern connection of a 3-web $S_{t_1}$, $S_{t_2}$, $S_{t_3}$ is a torsion-free affine holomorphic connection with holonomy in $GL(n, \mathbb{C})$ acting on $\mathbb{C}^{2n} = \mathbb{C}^n \otimes \mathbb{C}^2$, and independent from the choice of $t_i$. 
**$SL(2)$-webs on the space of quasilines**

**DEFINITION:** A quasiline on a complex manifold is an embedding $S \cong \mathbb{CP}^1 \rightarrow M$ such that the normal bundle $NS$ is isomorphic to $\mathcal{O}(1)^n$.

**THEOREM:** Let $W$ be a space of quasilines, $C \subset M$ a quasiline, $t \in \mathbb{CP}^1$, and $S_t(C) \subset TCW$ the space of all tangent vectors $v \in NC$ vanishing at $t \in C = \mathbb{CP}^1$. Consider a bundle $S_t \subset TW$ generated by $S_t(C)$ for all $C$. Then $S_t$ defines an $SL(2)$-web on $W$. ■

**EXAMPLE:** A line in $\mathbb{CP}^n$ is a quasiline.

**EXAMPLE:** A twistor space of a 4-dimensional Riemannian manifold with ASD metric contains quasilines.

**EXAMPLE:** A twistor space of a hyperkähler manifold contains quasilines.

**EXAMPLE:** Any Fano manifold is birational to a manifold containing quasilines.
Trisymplectic structure on a vector space

**DEFINITION:** A trisymplectic structure on a complex vector space of dimension $2n$ is a 3-dimensional space $\Omega \subset \Lambda^2 V$ of complex linear 2-forms, such that any $\eta \in \Omega$ has rank $2n$, $n$ or 0.

**REMARK:** It is easy to see that $\Omega$ contains a symplectic form.

**PROPOSITION:** Given two symplectic forms $\omega_1, \omega_2 \in \Omega$, consider the map $\varphi_{\Omega_1,\Omega_2} := \omega_1 \circ \omega_2^{-1} \in \text{End}(V)$. Then $\varphi_{\Omega_1,\Omega_2}$ can be expressed in an appropriate basis by the matrix

$$
\varphi_{\omega_1,\omega_2} = \begin{pmatrix}
\lambda & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & \lambda & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & \lambda & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \lambda' & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & \lambda' & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & \lambda'
\end{pmatrix},
$$

with the eigenspaces of equal dimension.

**THEOREM:** Let $(V, \Omega)$ be a trisymplectic vector space, and $H \subset \text{End}(V)$ the algebra generated by $\varphi_{\Omega_1,\Omega_2}$, for all $\omega_1, \omega_2 \in \Omega$. Then $H$ is isomorphic to the matrix algebra $\text{Mat}(2)$, acting on $V$ in a standard way.
Trisymplectic structures as Mat(2)-representations

**DEFINITION:** Let $V$ be a complex vector space with the **standard action** of the matrix algebra Mat(2), i.e. $V \cong V_0 \otimes \mathbb{C}^2$ and Mat(2) acts only through the second factor.

**REMARK:** Consider the natural $SL(2)$-action on $V$ induced by Mat(2), and extend it multiplicatively to all tensor powers of $V$. Let $g \in \text{Sym}_2^2(V)$ be an $SL(2)$-invariant, non-degenerate 2-form on $V$, and $\{I, J, K\}$ a quaternionic basis in Mat(2) Then

$$g(x, Iy) = g(Ix, I^2y) = -g(Ix, y)$$

hence the form $\Omega_I(\cdot, \cdot) := g(\cdot, I\cdot)$ is a symplectic form, obviously non-degenerate; the forms $\Omega_J, \Omega_K$ have the same properties. Let $\Omega := \langle \Omega_I, \Omega_J, \Omega_K \rangle$. It turns out that **this construction gives a trisymplectic structure, and all trisymplectic structures can be obtained in this way.**
**Trisymplectic structures as Mat(2)-representations II**

**THEOREM:** Let $V$ be a vector space equipped with a standard action of the matrix algebra $H \cong \text{Mat}(2)$, and $\{I, J, K\}$ a quaternionic basis in $\text{Mat}(2)$. Consider the corresponding action of $SL(2)$ on the tensor powers of $V$. Then, for any $SL(2)$-invariant symmetric form $g$, denote by $\Omega$ the space generated by $\Omega_I := g(\cdot, I \cdot)$, $\Omega_J$, $\Omega_K$. Then $\Omega$ is a trisymplectic structure on $V$, with the operators $\Omega_K^{-1} \circ \Omega_J$, $\Omega_K^{-1} \circ \Omega_I$ generating $H$. Moreover, for each trisymplectic structure $\Omega$ on $V$, there exists a unique (up to a constant) $SL(2)$-invariant non-degenerate quadratic form $g$ inducing $\Omega$ as above.
Trisymplectic manifold

**DEFINITION:** A trisymplectic structure on a complex $2n$-manifold $M$ is a triple of holomorphic symplectic forms $\Omega_1, \Omega_2, \Omega_3$, such that any linear combination of these forms has rank $2n$, $n$ or 0. We denote by $\Omega$ the 3-dimensional space generated by $\Omega_i$. Obviously, $\Omega$ defines a trisymplectic structure at each point of $M$.

**REMARK:** Let $\Omega_1, \Omega_2 \in \Omega$. Consider $P(t) := \det(\Omega_1 + t\Omega_2)$ as a polynomial of $t$. Since the eigenvalues of $\Omega_1 + t\Omega_2$ occur in $n$-tuples, $P(t) = Q(t)^{n/2}$, where $Q$ is a quadratic polynomial.

**CLAIM:** There exists a non-degenerate quadratic form $Q$ on $\Omega$, unique up to a constant, such that $\Omega \in \Omega$ is degenerate if and only if $Q(\Omega, \Omega) = 0$.

**COROLLARY:** For each degenerate $\Omega \in \Omega$, its radical $\ker \Omega$ is a sub-bundle of codimension $n$ in $TM$. Moreover, for all non-proportional degenerate $\Omega, \Omega' \in \Omega$, one has $TM = \ker \Omega \oplus \ker \Omega'$.

**REMARK:** Since $\Omega$ is closed, $\ker \Omega$ is involutive: $[\ker \Omega, \ker \Omega] \subset \ker \Omega$.

**REMARK:** Similar to web geometry!
Trisymplectic manifolds

**THEOREM:** For any trisymplectic structure on $M$, the bundles $\ker \Omega \subset TM$ define an $SL(2)$-web. Moreover, the Chern connection of this $SL(2)$-web preserves all forms in $\Omega$.

**REMARK:** In this case, the Chern connection has holonomy in $Sp(n, \mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.

**REMARK:** For a trisymplectic structure $\Omega$, it is just the Levi-Civita connection of the holomorphic Riemannian form associated with $\Omega$. 
Hyperkähler manifolds

**DEFINITION:** A hyperkähler structure on a manifold $M$ is a Riemannian structure $g$ and a triple of complex structures $I, J, K$, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that $g$ is Kähler for $I, J, K$.

**REMARK:** A hyperkähler manifold has three symplectic forms $\omega_I := g(I\cdot, \cdot)$, $\omega_J := g(J\cdot, \cdot)$, $\omega_K := g(K\cdot, \cdot)$.

**REMARK:** This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves $I, J, K$.

**DEFINITION:** Let $M$ be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called the holonomy group of $M$.

**REMARK:** A hyperkähler manifold can be defined as a manifold which has holonomy in $Sp(n)$ (the group of all endomorphisms preserving $I, J, K$).
Holomorphically symplectic manifolds

**DEFINITION:** A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic $(2,0)$-form.

**REMARK:** Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1} \omega_K$ is a holomorphic symplectic form on $(M,I)$.

**THEOREM:** (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

**EXAMPLE:** An even-dimensional complex torus.

**REMARK:** Take a symmetric square $\text{Sym}^2 T$, with a natural action of $T$, and let $T^{[2]}$ be a blow-up of a singular divisor. Then $T^{[2]}$ is naturally isomorphic to the Kummer surface $\widetilde{T}/\pm 1$.

**DEFINITION:** A K3 surface is a complex 2-manifold obtained as a deformation of a Kummer surface.

**REMARK:** A K3 surface is always hyperkähler. Any hyperkähler manifold of real dimension 4 is isomorphic to a torus or a K3 surface.
Complexification of a manifold

**Definition:** Let \( M \) be a complex manifold, equipped with an anticomplex involution \( \iota \). The fixed point set \( M_\mathbb{R} \) of \( \iota \) is called a real analytic manifold, and a germ of \( M \) in \( M_\mathbb{R} \) is called a complexification of \( M_\mathbb{R} \).

**Question:** What is a complexification of a Kähler manifold (considered as real analytic variety)?

**Theorem:** (D. Kaledin, B. Feix) Let \( M \) be a real analytic Kähler manifold, and \( M_\mathbb{C} \) its complexification. Then \( M_\mathbb{C} \) admits a hyperkähler structure, determined uniquely and functorially by the Kähler structure on \( M \).

**Question:** What is a complexification of a hyperkähler manifold?
Twistor space

**DEFINITION:** Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{ L := aI + bJ + cK, \quad a^2 + b^2 + c^2 = 1. \}$ They are usually non-algebraic. Indeed, if $M$ is compact, for generic $a, b, c$, $(M, L)$ has no divisors (Fujiki).

**DEFINITION:** A twistor space $\text{Tw}(M)$ of a hyperkähler manifold is a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$. More formally:

Let $\text{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on $M$ induced by $J \in S^2 \subset \mathbb{H}$. Let $I_J$ denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\text{Tw}} = I_m \oplus I_J : T_x \text{Tw}(M) \to T_x \text{Tw}(M)$ satisfies $I_{\text{Tw}}^2 = -\text{Id}$. It defines an almost complex structure on $\text{Tw}(M)$. This almost complex structure is known to be integrable (Obata).

**EXAMPLE:** If $M = \mathbb{H}^n$, $\text{Tw}(M) = \text{Tot} (\mathcal{O}(1) \oplus n) \cong \mathbb{C}P^{2n+1} \setminus \mathbb{C}P^{2n-1}$

**REMARK:** For $M$ compact, $\text{Tw}(M)$ never admits a Kähler structure.
Rational curves on $\text{Tw}(M)$.

**REMARK:** The twistor space has many rational curves. In fact, it is rationally connected (Campana).

**DEFINITION:** Denote by $\text{Sec}(M)$ the space of holomorphic sections of the twistor fibration $\text{Tw}(M) \xrightarrow{\pi} \mathbb{C}P^1$.

**DEFINITION:** For each point $m \in M$, one has a horizontal section $C_m := \{m\} \times \mathbb{C}P^1$ of $\pi$. The space of horizontal sections is denoted $\text{Sec}_{\text{hor}}(M) \subset \text{Sec}(M)$

**REMARK:** The space of horizontal sections of $\pi$ is identified with $M$. The normal bundle $NC_m = \mathcal{O}(1)^{\dim M}$. Therefore, some neighbourhood of $\text{Sec}_{\text{hor}}(M) \subset \text{Sec}(M)$ is a smooth manifold of dimension $2\dim M$.

**DEFINITION:** A twistor section $C \subset \text{Tw}(M)$ is called regular, if it is a quasiline: $NC = \mathcal{O}(1)^{\dim M}$.

**CLAIM:** For any $I \neq J \in \mathbb{C}P^n$, consider the evaluation map $\text{Sec}(M) \xrightarrow{E_{I,J}} (M, I) \times (M, J), s \mapsto s(I) \times s(J)$. Then $E_{I,J}$ is an isomorphism around the set $\text{Sec}_0(M)$ of regular sections.
Complexification of a hyperkähler manifold.

**REMARK:** Consider an anticomplex involution $\text{Tw}(M) \overset{\iota}{\longrightarrow} \text{Tw}(M)$ mapping $(m, t)$ to $(m, i(t))$, where $i : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ is a central symmetry. Then $\text{Sec}_{\text{hor}}(M) = M$ is a component of the fixed set of $\iota$.

**COROLLARY:** $\text{Sec}(M)$ is a complexification of $M$.

**QUESTION:** What are geometric structures on $\text{Sec}(M)$?

**Answer 1:** For compact $M$, $\text{Sec}(M)$ is holomorphically convex (Stein if $\dim M = 2$).

**Answer 2:** Let $I \in \mathbb{C}P^1$, and $ev_I : \text{Sec}_0(M) \longrightarrow (M, I)$ be an evaluation map putting $S \in \text{Sec}_0(M)$ to $S(I)$. Then the 2-forms $ev_I^* \Omega_I$, $I \in \mathbb{C}P^1$ generate a trisymplectic structure on $\text{Sec}_0(M)$.

**Answer 3:** The space $\text{Sec}_0(M)$ of quasilines admits a holomorphic, torsion-free connection with holonomy $Sp(n, \mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$. This is the Chern connection of the corresponding $SL(2)$-web.
**Mathematical instantons**

**DEFINITION:** A mathematical instanton on $\mathbb{CP}^3$ is a stable rank 2 bundle $B$ with $c_1(B) = 0$ and $H^1(B(-1)) = 0$. A framed instanton is a mathematical instanton equipped with a trivialization of $B|_\ell$ for some fixed line $\ell = \mathbb{CP}^1 \subset \mathbb{CP}^3$.

**REMARK:** The space $\mathcal{M}_c$ of framed instantons with $c_2 = c$ is a principal $\text{SL}(2)$-bundle over the space of all mathematical instantons trivial on $\ell$.

**DEFINITION:** An instanton on $\mathbb{CP}^2$ is a stable bundle $B$ with $c_1(B) = 0$. A framed instanton is an instanton equipped with a trivialization $B|x$ for some fixed point $x \in \mathbb{CP}^2$.

**THEOREM:** (Atiyah-Drinfeld-Hitchin-Manin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{CP}^2$ is smooth, connected, hyperkähler.

**THEOREM:** (Jardim–V.) The space $\mathcal{M}_c$ of framed mathematical instantons on $\mathbb{CP}^3$ is naturally identified with the space of twistor sections $\text{Sec}(\mathcal{M}_{2,c})$. 

16
The space of instantons on $\mathbb{C}P^3$

**CONJECTURE:** The space of mathematical instantons is smooth and connected.

**THEOREM:** (Grauert-Müllich, Hauzer-Langer) Every mathematical instanton on $\mathbb{C}P^3$ is trivial on some line $\ell \subset \mathbb{C}P^3$.

**COROLLARY:** The space of mathematical instantons is covered by Zariski open, dense subvarieties which take form $\mathbb{M}_c/SL(2,\mathbb{C})$.

**COROLLARY:** To prove that the space of mathematical instantons is smooth and connected it would suffice to prove it for $\mathbb{M}_c$.

**THEOREM:** (Jardim–V.) The space $\mathbb{M}_c$ is smooth.

**REMARK:** To prove that $\mathcal{M}_{r,c}$ is smooth, one could use hyperkähler reduction. To prove that $\mathbb{M}_c$ is smooth and connected, we develop trihyperkähler reduction, which is a trisymplectic reduction defined on the space of quasilines in a twistor space.

We prove that $\mathbb{M}_c$ is a trihyperkähler quotient of a vector space by a reductive group action, hence smooth.