3-webs and connections on the space of quasilines

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Holomorphic 3-webs.

DEFINITION: Let M be a complex manifold, and S_1 , S_2 , S_3 integrable, pairwise transversal holomorphic sub-bundles in TM, of dimension $\frac{1}{2} \dim M$. Then (S_1, S_2, S_3) is called a holomorphic 3-web on M.

REMARK: On smooth manifolds, the theory of 3-webs is due to Chern and Blaschke (1930-ies).

THEOREM: (Ph. D. thesis of Chern, 1936) Let S_1, S_2, S_3 be a holomorphic 3-web on a complex manifold M. Then there exists a unique holomorphic connection ∇ on M which preserves the sub-bundles S_i , and such that its torsion T satisfies $T(S_1, S_2) = 0$.

Holomorphic SL(2)-webs.

DEFINITION: A holomorphic 3-web S_1 , S_2 , S_3 on a complex manifold M is called an SL(2)-web if

• the projection operators $P_{i,j}$ of TM to S_i along S_j generate the standard action of Mat(2) on $\mathbb{C}^2 \otimes \mathbb{C}^n$,

• for any nilpotent $v \in Mat(2)$, the bundle $v(TM) \subset TM$ is involutive.

REMARK: The set of $v \in Mat(2)$ with $\operatorname{rk} v = 1$ satisfies $\mathbb{P}V = \mathbb{C}P^1$, hence the sub-bundles $v(TM) \subset TM$ are parametrized by $\mathbb{C}P^1$. An SL(2)-web is determined by a set of sub-bundles $S_t \subset TM$, $t \in \mathbb{C}P^1$, which are pairwise transversal and involutive.

THEOREM: Let $S_t \subset TM$, $t \in \mathbb{C}P^1$ be an SL(2)-web on M, and $t_1, t_2, t_3 \in \mathbb{C}P^1$ distinct points. Then the Chern connection of a 3-web S_{t_1} , S_{t_2} , S_{t_3} is a torsion-free affine holomorphic connection with holonomy in $GL(n,\mathbb{C})$ acting on $\mathbb{C}^{2n} = \mathbb{C}^n \otimes \mathbb{C}^2$, and independent from the choice of t_i .

SL(2)-webs on the space of quasilines

DEFINITION: A quasiline on a complex manifold is an embedding $S \cong \mathbb{C}P^1 \longrightarrow M$ such that the normal bundle NS is isomorphic to $\mathcal{O}(1)^n$.

THEOREM: Let W be a space of quasilines, $C \subset M$ a quasiline, $t \in \mathbb{C}P^1$, and $S_t(C) \subset T_C W$ the space of all tangent vectors $v \in NC$ vanishing at $t \in C = \mathbb{C}P^1$. Consider a bundle $S_t \subset TW$ generated by $S_t(C)$ for all C. **Then** S_t **defines an** SL(2)-web on W.

EXAMPLE: A line in $\mathbb{C}P^n$ is a quasiline.

EXAMPLE: A twistor space of a 4-dimensional Riemannian manifold with ASD metric contains quasilines.

EXAMPLE: A twistor space of a hyperkähler manifold contains quasilines.

EXAMPLE: Any Fano manifold is birational to a manifold containing quasilines.

Trisymplectic structure on a vector space

DEFINITION: A trisymplectic structure on a complex vector space of dimension 2n is a 3-dimensional space $\Omega \subset \Lambda^2 V$ of complex linear 2-forms, such that any $\eta \in \Omega$ has rank 2n, n or 0.

REMARK: It is easy to see that Ω contains a symplectic form.

PROPOSITION: Given two symplectic forms $\omega_1, \omega_2 \in \Omega$, consider the map $\varphi_{\Omega_1,\Omega_2} := \omega_1 \circ \omega_2^{-1} \in \text{End}(V)$. Then $\varphi_{\Omega_1,\Omega_2}$ can be expressed in an appropriate basis by the matrix

$$\varphi_{\omega_1,\omega_2} = \begin{pmatrix} \lambda & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \lambda & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda' & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda' & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \lambda' \end{pmatrix},$$

with the eigenspaces of equal dimension.

THEOREM: Let (V, Ω) be a be a trisymplectic vector space, and $H \subset \text{End}(V)$ the algebra generated by $\varphi_{\Omega_1,\Omega_2}$, for all $\omega_1,\omega_2 \in \Omega$. Then H is isomorphic to the matrix algebra Mat(2), acting on V in a standard way.

Trisymplectic structures as Mat(2)-**representations**

DEFINITION: Let V be a complex vector space with the standard action of the matrix algebra Mat(2), i.e. $V \cong V_0 \otimes \mathbb{C}^2$ and Mat(2) acts only through the second factor.

REMARK: Consider the natural SL(2)-action on V induced by Mat(2), and extend it multiplicatively to all tensor powers of V. Let $g \in \text{Sym}^2_{\mathbb{C}}(V)$ be an SL(2)-invariant, non-degenerate 2-form on V, and $\{I, J, K\}$ a quaternionic basis in Mat(2) Then

$$g(x, Iy) = g(Ix, I^2y) = -g(Ix, y)$$

hence the form $\Omega_I(\cdot, \cdot) := g(\cdot, I \cdot)$ is a symplectic form, obviously nondegenerate; the forms Ω_J , Ω_K have the same properties. Let $\Omega := \langle \Omega_I, \Omega_J, \Omega_K \rangle$. It turns out that this construction gives a trisymplectic structure, and all trisymplectic structures can be obtained in this way.

Trisymplectic structures as Mat(2)-representations II

THEOREM: Let *V* be a vector space equipped with a standard action of the matrix algebra $H \cong Mat(2)$, and $\{I, J, K\}$ a quaternionic basis in Mat(2). Consider the corresponding action of SL(2) on the tensor powers of *V*. Then, for any SL(2)-invariant symmetric form *g*, denote by Ω the space generated by $\Omega_I := g(\cdot, I \cdot), \ \Omega_J, \ \Omega_K$ Then Ω is a trisymplectic structure on *V*, with the operators $\Omega_K^{-1} \circ \Omega_J, \ \Omega_K^{-1} \circ \Omega_I$ generating *H*. Moreover, for each trisymplectic structure Ω on *V*, there exists a unique (up to a constant) SL(2)-invariant non-degenerate quadratic form *g* inducing Ω as above.

Trisymplectic manifold

DEFINITION: A trisymplectic structure on a complex 2n-manifold M is a triple of holomorphic symplectic forms Ω_1 , Ω_2 , Ω_3 , such that any linear combination of these forms has rank 2n, n or 0. We denote by Ω the 3-dimensional space generated by Ω_i . Obviously, Ω defines a trisymplectic structure at each point of M.

REMARK: Let $\Omega_1, \Omega_2 \in \Omega$. Consider $P(t) := \det(\Omega_1 + t\Omega_2)$ as a polynomial of t. Since the eigenvalues of $\Omega_1 + t\Omega_2$ occur in n-tuples, $P(t) = Q(t)^{n/2}$, where Q is a quadratic polynomial.

CLAIM: There exists a non-degenerate quadratic form Q on Ω , unique up to a constant, such that $\Omega \in \Omega$ is degenerate if and only if $Q(\Omega, \Omega) = 0$.

COROLLARY: For each degenerate $\Omega \in \Omega$, its radical ker Ω is a subbundle of codimension n in TM. Moreover, for all non-proportional degenerate $\Omega, \Omega' \in \Omega$, one has $TM = \ker \Omega \oplus \ker \Omega'$.

REMARK: Since Ω is closed, ker Ω is **involutive:** [ker Ω , ker Ω] \subset ker Ω .

REMARK: Similar to web geometry!

Trisymplectic manifolds

THEOREM: For any trisymplectic structure on M, the bundles ker $\Omega \subset TM$ define an SL(2)-web. Moreover, the Chern connection of this SL(2)-web preserves all forms in Ω .

REMARK: In this case, the Chern connection has holonomy in $Sp(n, \mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$.

REMARK: For a trisymplectic structure Ω , it is just the Levi-Civita connection of the holomorphic Riemannian form associated with Ω .

Hyperkähler manifolds

DEFINITION: A hyperkähler structure on a manifold M is a Riemannian structure g and a triple of complex structures I, J, K, satisfying quaternionic relations $I \circ J = -J \circ I = K$, such that g is Kähler for I, J, K.

REMARK: A hyperkähler manifold has three symplectic forms $\omega_I := g(I \cdot, \cdot), \ \omega_J := g(J \cdot, \cdot), \ \omega_K := g(K \cdot, \cdot).$

REMARK: This is equivalent to $\nabla I = \nabla J = \nabla K = 0$: the parallel translation along the connection preserves I, J, K.

DEFINITION: Let M be a Riemannian manifold, $x \in M$ a point. The subgroup of $GL(T_xM)$ generated by parallel translations (along all paths) is called **the holonomy group** of M.

REMARK: A hyperkähler manifold can be defined as a manifold which has holonomy in Sp(n) (the group of all endomorphisms preserving I, J, K).

Holomorphically symplectic manifolds

DEFINITION: A holomorphically symplectic manifold is a complex manifold equipped with non-degenerate, holomorphic (2,0)-form.

REMARK: Hyperkähler manifolds are holomorphically symplectic. Indeed, $\Omega := \omega_J + \sqrt{-1} \omega_K$ is a holomorphic symplectic form on (M, I).

THEOREM: (Calabi-Yau) A compact, Kähler, holomorphically symplectic manifold admits a unique hyperkähler metric in any Kähler class.

EXAMPLE: An even-dimensional complex torus.

REMARK: Take a symmetric square Sym² T, with a natural action of T, and let $T^{[2]}$ be a blow-up of a singular divisor. Then $T^{[2]}$ is naturally isomorphic to the Kummer surface $T/\pm 1$.

DEFINITION: A K3 surface is a complex 2-manifold obtained as a deformation of a Kummer surface.

REMARK: A K3 surface is always hyperkähler. Any hyperkähler manifold of real dimension 4 is isomorphic to a torus or a K3 surface.

Complexification of a manifold

DEFINITION: Let M be a complex manifold, equipped with an anticomplex involution ι . The fixed point set $M_{\mathbb{R}}$ of ι is called **a real analytic manifold**, and a germ of M in $M_{\mathbb{R}}$ is called **a complexification** of $M_{\mathbb{R}}$.

QUESTION: What is a complexification of a Kähler manifold (considered as real analytic variety)?

THEOREM: (D. Kaledin, B. Feix) Let M be a real analytic Kähler manifold, and $M_{\mathbb{C}}$ its complexification. Then $M_{\mathbb{C}}$ admits a hyperkähler structure ture, determined uniquely and functorially by the Kähler structure on M.

QUESTION: What is a complexification of a hyperkähler manifold?

Twistor space

DEFINITION: Induced complex structures on a hyperkähler manifold are complex structures of form $S^2 \cong \{L := aI + bJ + cK, a^2 + b^2 + c^2 = 1.\}$ **They are usually non-algebraic**. Indeed, if *M* is compact, for generic *a*, *b*, *c*, (*M*, *L*) has no divisors (Fujiki).

DEFINITION: A twistor space Tw(M) of a hyperkähler manifold is a complex manifold obtained by gluing these complex structures into a holomorphic family over $\mathbb{C}P^1$. More formally:

Let $\mathsf{Tw}(M) := M \times S^2$. Consider the complex structure $I_m : T_m M \to T_m M$ on M induced by $J \in S^2 \subset \mathbb{H}$. Let I_J denote the complex structure on $S^2 = \mathbb{C}P^1$.

The operator $I_{\mathsf{T}\mathsf{W}} = I_m \oplus I_J : T_x \mathsf{T}\mathsf{W}(M) \to T_x \mathsf{T}\mathsf{W}(M)$ satisfies $I^2_{\mathsf{T}\mathsf{W}} = -\mathrm{Id}$. It defines an almost complex structure on $\mathsf{T}\mathsf{W}(M)$. This almost complex structure is known to be integrable (Obata).

EXAMPLE: If
$$M = \mathbb{H}^n$$
, $\mathsf{Tw}(M) = \mathsf{Tot}(\mathcal{O}(1)^{\oplus n}) \cong \mathbb{C}P^{2n+1} \setminus \mathbb{C}P^{2n-1}$

REMARK: For *M* compact, Tw(M) never admits a Kähler structure.

Rational curves on Tw(M).

REMARK: The twistor space has many rational curves. In fact, it is rationally connected (Campana).

DEFINITION: Denote by Sec(M) the space of holomorphic sections of the twistor fibration $Tw(M) \xrightarrow{\pi} \mathbb{C}P^1$.

DEFINITION: For each point $m \in M$, one has a horizontal section $C_m := \{m\} \times \mathbb{C}P^1$ of π . The space of horizontal sections is denoted $Sec_{hor}(M) \subset Sec(M)$

REMARK: The space of horizontal sections of π is identified with M. The normal bundle $NC_m = \mathcal{O}(1)^{\dim M}$. Therefore, **some neighbourhood** of $Sec_{hor}(M) \subset Sec(M)$ is a smooth manifold of dimension $2 \dim M$.

DEFINITION: A twistor section $C \subset \mathsf{Tw}(M)$ is called **regular**, if it is a quasiline: $NC = \mathcal{O}(1)^{\dim M}$.

CLAIM: For any $I \neq J \in \mathbb{C}P^n$, consider the evaluation map $Sec(M) \xrightarrow{E_{I,J}} (M,I) \times (M,J)$, $s \longrightarrow s(I) \times s(J)$. Then $E_{I,J}$ is an isomorphism around the set $Sec_0(M)$ of regular sections.

Complexification of a hyperkähler manifold.

REMARK: Consider an anticomplex involution $\mathsf{Tw}(M) \xrightarrow{\iota} \mathsf{Tw}(M)$ mapping (m,t) to (m,i(t)), where $i : \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1$ is a central symmetry. Then $\mathsf{Sec}_{hor}(M) = M$ is a component of the fixed set of ι .

COROLLARY: Sec(M) is a complexification of M.

QUESTION: What are geometric structures on Sec(M)?

Answer 1: For compact M, Sec(M) is holomorphically convex (Stein if dim M = 2).

Answer 2: Let $I \in \mathbb{C}P^1$, and ev_I : $Sec_0(M) \longrightarrow (M, I)$ be an evaluation map putting $S \in Sec_0(M)$ to S(I). Then **the 2-forms** $ev_I^*\Omega_I$, $I \in \mathbb{C}P^1$ **generate a trisymplectic structure on** $Sec_0(M)$.

Answer 3: The space $Sec_0(M)$ of quasilines admits a holomorphic, torsion-free connection with holonomy $Sp(n, \mathbb{C})$ acting on $\mathbb{C}^{2n} \otimes \mathbb{C}^2$. This is the Chern connection of the corresponding SL(2)-web.

Mathematical instantons

DEFINITION: A mathematical instanton on $\mathbb{C}P^3$ is a stable rank 2 bundle *B* with $c_1(B) = 0$ and $H^1(B(-1)) = 0$. **A framed instanton** is a mathematical instanton equipped with a trivialization of $B|_{\ell}$ for some fixed line $\ell = \mathbb{C}P^1 \subset \mathbb{C}P^3$.

REMARK: The space \mathbb{M}_c of framed instantons with $c_2 = c$ is a principal SL(2)-bundle over the space of all mathematical instantons trivial on ℓ .

DEFINITION: An instanton on $\mathbb{C}P^2$ is a stable bundle *B* with $c_1(B) = 0$. A framed instanton is an instanton equipped with a trivialization $B|_x$ for some fixed point $x \in \mathbb{C}P^2$.

THEOREM: (Atiyah-Drinfeld-Hitchin-Manin) The space $\mathcal{M}_{r,c}$ of framed instantons on $\mathbb{C}P^2$ is **smooth**, **connected**, **hyperkähler**.

THEOREM: (Jardim–V.) The space \mathbb{M}_c of framed mathematical instantons on $\mathbb{C}P^3$ is naturally identified with the space of twistor sections $Sec(\mathcal{M}_{2,c})$.

The space of instantons on $\mathbb{C}P^3$

CONJECTURE: The space of mathematical instantons is smooth and connected.

THEOREM: (Grauert-Müllich, Hauzer-Langer) **Every mathematical in**stanton on $\mathbb{C}P^3$ is trivial on some line $\ell \subset \mathbb{C}P^3$.

COROLLARY: The space of mathematical instantons is covered by **Zariski open, dense subvarieties** which take form $\mathbb{M}_c/SL(2,\mathbb{C})$.

COROLLARY: To prove that the space of mathematical instantons is smooth and connected it would suffice to prove it for M_c .

THEOREM: (Jardim–V.) **The space** M_c is smooth.

REMARK: To prove that $\mathcal{M}_{r,c}$ is smooth, one could use hyperkähler reduction. To prove that \mathbb{M}_c is smooth and connected, we develop **tri-hyperkähler reduction**, which is a **trisymplectic reduction defined on the space of quasilines in a twistor space**.

We prove that M_c is a trihyperkähler quotient of a vector space by a reductive group action, hence smooth.